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On subordination of holomorphic semigroups ☆

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We prove that for any Bernstein function ψ the operator $-\psi(A)$ generates a bounded holomorphic C_0-semigroup $(e^{-t\psi(A)})_{t\geq 0}$ on a Banach space, whenever $-A$ does. This answers a question posed by Kishimoto and Robinson. Moreover, giving a positive answer to a question by Berg, Boyadzhiev and de Laubenfels, we show that $(e^{-t\psi(A)})_{t\geq 0}$ is holomorphic in the holomorphy sector of $(e^{-tA})_{t\geq 0}$, and if $(e^{-tA})_{t\geq 0}$ is sectorially bounded in this sector then $(e^{-t\psi(A)})_{t\geq 0}$ has the same property. We also obtain new sufficient conditions on ψ in order that, for every Banach space X, the semigroup $(e^{-t\psi(A)})_{t\geq 0}$ on X is holomorphic whenever $(e^{-tA})_{t\geq 0}$ is a bounded C_0-semigroup on X. These conditions improve and generalize well-known results by Carasso–Kato and Fujita.

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1. Introduction

The present paper concerns operator-theoretic and function-theoretic properties of Bernstein functions and solves several notable problems which have been left open for some time.

Bernstein functions play a prominent role in probability theory and operator theory. One of their characterizations, also important for our purposes, says that a function $\psi : (0, \infty) \to [0, \infty)$ is Bernstein if and only if there exists a vaguely continuous semigroup of subprobability Borel measures $(\mu_t)_{t\geq 0}$ on $[0, \infty)$ such that

$$e^{-t\psi(\lambda)} = \int_0^\infty e^{-\lambda s}\, \mu_t(ds), \qquad \lambda > 0, \tag{1.1}$$

for all $t \geq 0$.

Let now $(e^{-tA})_{t\geq 0}$ be a bounded C_0-semigroup on a (complex) Banach space X with generator $-A$. The relation (1.1) suggests a way to define a new bounded C_0-semigroup $(e^{-tB})_{t\geq 0}$ on X in terms of $(e^{-tA})_{t\geq 0}$ and a Bernstein function ψ as

$$e^{-tB} = \int_0^\infty e^{-sA}\, \mu_t(ds), \tag{1.2}$$

where $(\mu_t)_{t\geq 0}$ is a semigroup of measures given by (1.1). Following (1.1), it is natural to define $\psi(A) := B$. As it will be revealed in Subsection 3.3 below, such a definition of $\psi(A)$ goes far beyond formal notation and it respects some rules for operator functions called functional calculus.

The semigroup $(e^{-t\psi(A)})_{t\geq 0}$ is subordinated to the semigroup $(e^{-tA})_{t\geq 0}$ via a subordinator $(\mu_t)_{t\geq 0}$. The basics of subordination theory was set up by Bochner [4] and Phillips [27]. This approach to constructing semigroups is motivated by probabilistic applications, e.g. by the study of Lévy processes, but it has also significant value for PDEs as well. As a textbook example one may mention a classical result of Yosida expressing $(e^{-tA^\alpha})_{t\geq 0}$, $\alpha \in (0, 1)$, in terms of $(e^{-tA})_{t\geq 0}$ as in (1.2), see e.g. [33]. The essential feature of this example is that C_0-semigroups $(e^{-tA^\alpha})_{t\geq 0}$ turn out to be necessarily holomorphic. This fact stimulated further research on relations between functional calculi and Bernstein functions, see e.g. [31] and [32]. Some of them are described below.

An easy consequence of (1.2) is that for a fixed Bernstein function ψ the mapping

$$\mathcal{M} : -A \mapsto -\psi(A) \tag{1.3}$$

preserves the class of generators of bounded C_0-semigroups, and it is natural to ask whether there are any other important classes of semigroup generators stable under $\mathcal{M}$. In particular, whether $\mathcal{M}$ preserves the class of holomorphic C_0-semigroups. The question