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Critical weak immersed surfaces within sub-manifolds of the Teichmüller space

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ABSTRACT

We prove that the critical points of various energies such as the area, the Willmore energy, the frame energy for tori, etc., among possibly branched immersions constrained to evolve within a smooth sub-manifold of the Teichmüller space satisfy the corresponding constrained Euler Lagrange equation. We deduce that critical points of the Willmore energy or the frame energy for tori are smooth analytic surfaces, away possibly from isolated branched points, under the condition that either the genus is at most 2 or if the sub-manifold does not intersect the subspace of hyper-elliptic points. Using a compactness result from [19] we can conclude that each closed sub-manifold of the Teichmüller space, including points, under the previous assumptions, possess a possibly branched smooth Willmore minimizer satisfying the conformal-constrained Willmore equation.

1. Introduction

The purpose of the present work is to derive constrained Euler Lagrange equations for weak immersions which are critical points of geometric energies such as the area, the Willmore energy, the frame energy for tori, etc., under the constraint that the metrics

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defined by the variation of this immersions stay within a given sub-manifold of the Teichmüller space.

The study of the variations of Willmore energy has been initiated by Leon Simon in a seminal work [20] in which he was proving the existence of an embedded torus into $\mathbb{R}^m$ minimizing the L^2 norm of the second fundamental form. This problem was an analytical challenge at the time in particular due to the fact that this norm clearly does not provide any control to the C^1 norm of the surface which is needed in order to speak about immersion. The need of weakening the strict geometric notion of immersion was answered in this work by considering "measure theoretic version" of sub-manifold known as *varifold* and by using local approximation of these weak objects by *bi-harmonic graphs*. In the following decade, after this analytical breakthrough, a series of works ([2,7–9], etc.) took over successfully this approach to solve important questions related to Willmore surfaces such as the existence of the flow, point removability property for Willmore surfaces, existence of Willmore minimizers within a given conformal class, etc.

In [16,16] the author introduced a *parametric approach* to the study of Willmore Lagrangian and the notion of *weak immersions*. In these works one studies surfaces in $\mathbb{R}^m$ from the point of view of the immersion, the map which is generating them, while the Leon Simon's approach is mostly considering the immersed surface as a subset of the ambient space $\mathbb{R}^m$ and is called the *ambient approach* to the Willmore problem.

We now recall the notion of *weak immersion* with L^2-bounded second fundamental form introduced in [19].

A closed abstract surface Σ being given, we observe that any smooth Riemannian metric is equivalent to any other one. One can then define the Sobolev spaces $W^{k,p}_{g_0}(\Sigma,\mathbb{R}^m)$ from Σ into $\mathbb{R}^m$, for $k\in\mathbb{N}$ and $p\in[1,\infty]$ with respect to any smooth reference metric g_0 and all these spaces are independent of the chosen metric g_0 and we can simply denote them $W^{k,p}(\Sigma,\mathbb{R}^m)$.

A *weak immersion* of Σ into $\mathbb{R}^m$ is a map $\vec{\Phi}$ from Σ into $\mathbb{R}^m$ such that

i)

$$\vec{\Phi}\in W^{1,\infty}\cap W^{2,2}(\Sigma,\mathbb{R}^m)\quad ,$$

ii) there exists a constant $C_{\vec{\Phi}}>1$ such that

$$\forall x\in\Sigma\quad\forall X\in T_x\Sigma\qquad C_{\vec{\Phi}}^{-1}\ g_0(X)\le|d\vec{\Phi}(X)|^2\le C_{\vec{\Phi}}\ g_0(X)$$

i.e. in other words the metric on Σ equal to the pull back by $\vec{\Phi}$ of the canonical metric of $\mathbb{R}^m$ is equivalent to any reference metric on Σ.

Observe that for a weak immersion

$$\vec{n}_{\vec{\Phi}}\in W^{1,2}(\Sigma,\wedge^{m-2}\mathbb{R}^m)$$