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Spectrality of a class of infinite convolutions ☆

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ABSTRACT

For a finite set $\mathcal{D} \subset \mathbb{Z}$ and an integer $b \geq 2$, we say that $(b, \mathcal{D})$ is compatible with $\mathcal{C} \subset \mathbb{Z}$ if $[e^{-2\pi i dc/b}]_{d\in\mathcal{D},c\in\mathcal{C}}$ is a Hadamard matrix. Let $\delta_E = \frac{1}{\#E}\sum_{a\in E}\delta_a$ denote the uniformly discrete probability measure on E. We prove that the class of infinite convolution (Moran measure) $\mu_{b,\{\mathcal{D}_k\}} = \delta_{b^{-1}\mathcal{D}_1} * \delta_{b^{-2}\mathcal{D}_2} * \cdots$ is a spectral measure provided that there is a common $\mathcal{C} \subset \mathbb{Z}^+$ compatible to all the $(b, \mathcal{D}_k)$ and $\mathcal{C} + \mathcal{C} \subseteq \{0, 1, \ldots, b-1\}$. We also give some examples to illustrate the hypotheses and results, in particular, the last condition on $\mathcal{C}$ is essential.

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1. Introduction

Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support. We call a countable subset $\Lambda \subset \mathbb{R}^d$ a *spectrum* of μ if the set of complex exponentials $\mathbb{E}_\Lambda = \{e_\lambda : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$, where $e_\lambda(x) = e^{-2\pi i\langle x,\lambda\rangle}$. If $L^2(\mu)$ admits a spectrum, then μ is called a *spectral measure*; in this case, μ is of pure type, i.e., it is either discrete and finite, or absolutely continuous or singular continuous with respect to the Lebesgue measure [14]. In particular, if μ is the normalized Lebesgue measure supported on a Borel set Ω, then Ω is called a *spectral set*. The interest of the spectral set was first raised by Fuglede [13] in 1974, who conjectured that Ω *is a spectral set if and only if it is a translational tile.*

Fuglede's conjecture was disproved in 2004 by Tao [30] who showed that a spectral set is not necessary a tile for $\mathbb{R}^d, d \geq 5$. Tao's work was subsequently improved by Kolountzakis and Matolcsi [17,18] who proved that the conjecture is false in both directions for $d \geq 3$. Nevertheless, there are positive results with additional conditions on the underlying sets, and the conjecture is still valid for $\mathbb{R}^d$, $d \leq 2$. In another direction, Jorgensen and Pedersen [16] made a head start to study the spectral property of the self-similar measures. Nowadays, there is a large literature on this topic [1–7,9,12–16,19–21,23–26,28–30]. Among those, one of the best known results is that if $\rho = 1/q$ for some integer $q > 1$, then $\mu_{\rho,\{0,1\}}$ is a spectral measure if and only if q is an even integer [16]. In [2], Dai characterized that these $\mu_{1/q,\{0,1\}}$ are the only spectral measures among the Bernoulli convolutions.

Until now, there are only a few classes of singular spectral measures that are known. In this paper, we will consider a new class of singular spectral measures defined by some simple conditions. More specifically, we will study the *Moran measures*, which is a non-self-similar extension of the Cantor measure (as well as the Bernoulli convolution) through the infinite convolution. For any finite set, denote $\delta_E = \frac{1}{\#E}\sum_{a\in E}\delta_a$, where $\#E$ is the cardinality of the set E and δ_a is the Dirac point mass measure at a. Let $\{b_k\}_{k=1}^{\infty}$ be a sequence of integers > 1, and let $\{\mathcal{D}_k\}_{k=1}^{\infty}$ be a sequence of finite *digit sets* in $\mathbb{Z}$. If $\sup_{k\geq 1}|\mathcal{D}_k| := \sup\{\ |d| : d \in \mathcal{D}_k, k \geq 1\} < \infty$, then associated to these two sequences, there exists a Borel probability measure with compact support defined by the convolution

$$\mu_{\{b_k\},\{\mathcal{D}_k\}} = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1b_2)^{-1}\mathcal{D}_2} * \cdots$$