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Gap vectors of real projective varieties

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ABSTRACT

Let $X \subseteq \mathbb{P}^m$ be a totally real, non-degenerate, projective variety and let $\Gamma \subseteq X(\mathbb{R})$ be a generic set of points. Let P be the cone of nonnegative quadratic forms on X and let Σ be the cone of sums of squares of linear forms. We examine the dimensions of the faces $P(\Gamma)$ and $\Sigma(\Gamma)$ consisting of forms in P and Σ, which vanish on Γ. As the cardinality of the set Γ varies in $1, 2, \ldots, \text{codim}(X)$, the difference between the dimensions of $P(\Gamma)$ and $\Sigma(\Gamma)$ defines a numerical invariant of X, which we call the gap vector of X. Our main result is a formula relating the components of the gap vector of X and the quadratic deficiencies of X and its generic projections. Using it, we prove that gap vectors are weakly increasing, obtain upper bounds for their rate of growth and prove that these upper bounds are eventually achieved for all varieties. Moreover, we give a characterization of the varieties with the simplest gap vectors: we prove that the gap vector vanishes identically precisely for varieties of minimal degree, giving another proof that $P \neq \Sigma$ when X is not a variety of minimal

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degree [4]. We also characterize the varieties whose gap vector equals $(0, \dots, 0, 1)$.

1. Introduction

Let $X \subseteq \mathbb{P}^m$ be a totally real, non-degenerate, projective variety with homogeneous coordinate ring R. The vector space R_2 of real quadratic forms on X contains two natural convex cones: the cone P of nonnegative quadratic forms and the cone Σ of sums of squares of linear forms. The relationship between these two cones has been an object of much interest since the time of Hilbert [14]. Recent results (see [1] and [4]) suggest that this relationship is in fact controlled by algebro-geometric features of the variety X over $\mathbb{C}$. The study of this relationship is arguably the central problem of the emerging field of *Convex Algebraic Geometry* [3]. The purpose of this article is to contribute to this line of research by asking: What geometric features of X control the dimensions of "generic" exposed faces of P and Σ? To more precisely formulate this question we introduce the gap vector, a new numerical invariant of real projective varieties.

Definition 1.1. For a set $\Gamma \subseteq X(\mathbb{R})$, let $\Sigma(\Gamma)$ and $P(\Gamma)$ consist of forms in Σ and P, which vanish on Γ. For $1 \leq j \leq c := \operatorname{codim}(X)$, we define $g_j(X) := \dim(P(\Gamma)) - \dim(\Sigma(\Gamma))$, where $\Gamma \subseteq X(\mathbb{R})$ is a generic set of points of cardinality j. These numbers are arranged in the gap vector of X, denoted by $g(X) := (g_1(X), \dots, g_c(X))$.

The purpose of the article is to characterize the fundamental properties of the gap vector. We demonstrate that this invariant is determined by the geometry of the variety X over $\mathbb{C}$ and that it has strong connections to classical algebraic geometry including the dimensions of secant varieties, the geometry of generic projections, and the quadratic deficiency of a variety.

Note that the sets $P(\Gamma)$ and $\Sigma(\Gamma)$ are exposed faces of the cones P and Γ, respectively, and a non-vanishing gap vector gives a certificate for strict inclusion of Σ in P. Using dimensional differences to certify this strict inclusion has a long history starting with Hilbert's original paper [14] on globally nonnegative polynomials, which launched the investigation of connections between nonnegativity and sums of squares. The dimensional difference aspect of Hilbert's approach was made clear by a modern exposition of Hilbert's method due to Reznick [16]. We note that globally nonnegative forms of degree $2d$ correspond to the special case where $X = \nu_d(\mathbb{P}^n)$ is the dth Veronese embedding of $\mathbb{P}^n$. For this situation dimensional differences between P and Σ were studied by the first three authors in [2]. One of the objectives of the present article is to refine and generalize these results to projective varieties.

There are several advantages of working with arbitrary projective varieties. First, to understand nonnegative forms and sums of squares of degree $2d$ on a variety X, it suffices