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Slicing inequalities for measures of convex bodies

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ABSTRACT

We consider the following problem. Does there exist an absolute constant C so that for every $n \in \mathbb{N}$, every integer $1 \leq k < n$, every origin-symmetric convex body L in $\mathbb{R}^n$, and every measure μ with non-negative even continuous density in $\mathbb{R}^n$,

$$\mu(L) \ \leq \ C^k \max_{H \in Gr_{n-k}} \mu(L \cap H) \ |L|^{k/n}, \tag{1}$$

where Gr_{n-k} is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and $|L|$ stands for volume? This question is an extension to arbitrary measures (in place of volume) and to sections of arbitrary codimension k of the slicing problem, a major open problem in convex geometry.

It was proved in [18,19] that (1) holds for arbitrary origin-symmetric convex bodies, all k and all μ with $C \leq O(\sqrt{n})$. In this article, we prove inequality (1) with an absolute constant C for unconditional convex bodies and for duals of bodies with bounded volume ratio. We also prove that for every $\lambda \in (0, 1)$ there exists a constant $C = C(\lambda)$ so that inequality (1) holds for every $n \in \mathbb{N}$, every origin-symmetric convex body L in $\mathbb{R}^n$, every measure μ with continuous density and the codimension of sections $k \geq \lambda n$. The proofs are based on a stability result for generalized intersection bodies and on estimates of the outer volume ratio distance from an arbitrary convex body to the classes of generalized intersection bodies. In the last section, we show that for some measures the behavior of minimal sections may be very different from the case of volume.

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1. Introduction

The slicing problem [4,5,1,32], a major open problem in convex geometry, asks whether there exists an absolute constant C so that for any origin-symmetric convex body K in $\mathbb{R}^n$ of volume 1 there is a hyperplane section of K whose $(n-1)$-dimensional volume is greater than $1/C$. In other words, does there exist a constant C so that for any $n \in \mathbb{N}$ and any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \le C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|, \tag{2}$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to ξ, and $|K|$ stands for volume of proper dimension? The best current result $C \le O(n^{1/4})$ is due to Klartag [13], who removed the logarithmic term from an earlier estimate of Bourgain [6]. We refer the reader to [8] for the history and partial results.

For certain classes of bodies the question has been answered in affirmative. These classes include unconditional convex bodies (as initially observed by Bourgain; see also [32,12,3,8]), unit balls of subspaces of L_p [2,11,30], intersection bodies [9, Theorem 9.4.11], zonoids, duals of bodies with bounded volume ratio [32], the Schatten classes [26], k-intersection bodies [23,20].

Iterating (2) one gets the lower dimensional slicing problem asking whether the inequality

$$|K|^{\frac{n-k}{n}} \le C^k \max_{H \in Gr_{n-k}} |K \cap H| \tag{3}$$

holds with an absolute constant C where $1 \le k \le n-1$ and Gr_{n-k} is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$.

In this note we prove (3) in the case where $k \ge \lambda n$, $0 < \lambda < 1$, with the constant $C = C(\lambda)$ dependent only on λ. Moreover, we prove this result in a more general setting of arbitrary measures in place of volume. We consider the following generalization of the slicing problem.

Problem 1. Does there exist an absolute constant C so that for every $n \in \mathbb{N}$, every integer $1 \le k < n$, every origin-symmetric convex body L in $\mathbb{R}^n$, and every measure μ with non-negative even continuous density f in $\mathbb{R}^n$,

$$\mu(L) \;\le\; C^k \max_{H \in Gr_{n-k}} \mu(L \cap H)\; |L|^{k/n}. \tag{4}$$

Here $\mu(B) = \int_B f$ for every compact set B in $\mathbb{R}^n$, and $\mu(B \cap H) = \int_{B\cap H} f$ is the result of integration of the restriction of f to H with respect to Lebesgue measure in H.

In many cases we will write (4) in an equivalent form

$$\mu(L) \;\le\; C^k \frac{n}{n-k}\, c_{n,k} \max_{H \in Gr_{n-k}} \mu(L \cap H)\; |L|^{k/n}, \tag{5}$$