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Explicit integral criteria for the existence of positive solutions of the linear delayed equation $\dot{x}(t) = -c(t)x(t - \tau(t))$

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ABSTRACT

The paper analyzes the linear differential equation with single delay $\dot{x}(t) = -c(t)x(t - \tau(t))$ with continuous $\tau\colon [t_0, \infty) \to (0, r]$, $r > 0$, $t_0 \in \mathbb{R}$, and $c\colon [t_0 - r, \infty) \to (0, \infty)$. New explicit integral criteria for the existence of a positive solution expressed in terms of c and τ are derived, an overview of known relevant criteria is provided, and relevant comparisons are also given. It is demonstrated that the known criteria are consequences of the new results.

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1. Introduction

In the paper we consider the equation

$$\dot{x}(t) = -c(t)x(t - \tau(t)) \tag{1}$$

where $\tau\colon [t_0, \infty) \to (0, r]$ is a continuous function, $t_0 \in \mathbb{R}$, $0 < r = \text{const}$, $c\colon [t_0 - r, \infty) \to \mathbb{R}^+$, $\mathbb{R}^+ := (0, \infty)$ is a continuous function, and the symbol " $\dot{}$ " stands for the right-hand derivative.

A function $x\colon [t^* - r, \infty) \to \mathbb{R}$ is called a *solution* of (1) corresponding to an initial point $t^* \in [t_0, \infty)$ if x is defined and continuous on $[t^* - r, \infty)$, differentiable on $[t^*, \infty)$, and satisfies (1) for $t \geq t^*$. We denote by $x(t^*, \varphi)(t)$ a solution of (1) corresponding to an initial point $t^* \in [t_0, \infty)$ generated by a continuous initial function $\varphi\colon [t^* - r, t^*] \to \mathbb{R}$. In the case of a linear equation (1), the solution $x(t^*, \varphi)(t)$ is unique on its maximal existence interval $[t^*, \infty)$ (see, e.g., [25]). Implicitly, we assume that the initial point equals t_0 unless another point is mentioned.

As is customary, a solution of (1) corresponding to an initial point $t^* \in [t_0, \infty)$ is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *non-oscillatory*. A non-oscillatory solution x of (1) corresponding to an initial point t^* is called *positive* (*negative*) if $x(t) > 0$ ($x(t) < 0$) on $[t^* - r, \infty)$. A non-oscillatory solution x of (1) corresponding to an initial point t^* is called *eventually positive* (*eventually negative*) if there exists $t^{**} > t^*$ such that $x(t) > 0$ ($x(t) < 0$) on $[t^{**}, \infty)$. Instead of the terms "eventually positive" ("eventually negative"), the phrase "positive for $t \to \infty$" ("negative for $t \to \infty$") is very often used.

In the paper, we derive (in Section 2) new integral criteria for the existence of a positive solution of the scalar differential equation with delay (1). Moreover, we give an upper bound for a class of positive solutions. Our criteria are more general than an explicit criterion derived recently in [4]. We compare both results in Section 2 and an overview is given of other results together with their comparison with a new result in Section 3. It is demonstrated that the known criteria are consequences of the new results. A proof of the new criterion (Theorem 3 below) is given in Section 4. Two examples are considered in Section 5. Some open problems are presented in Section 6.

2. Explicit integral criteria

In the literature, one can find several implicit criteria for the existence of positive solutions of linear delayed equations. We refer, e.g., to monographs [1,2,21,22,24], papers [7,14,16,15] and to the references therein.

Now we give an implicit theorem on the existence of a positive solution (see, e.g. [23, Theorem 1, Assertion 7 and Corollary 2.1] and [21, Theorem 2.1.4.]) with formulation adapted to Eq. (1).