

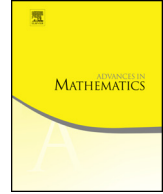


ELSEVIER

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim



# A pointwise inequality for fractional laplacians



Antonio Córdoba, Ángel D. Martínez\*

*Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Departamento de Matemáticas (Universidad Autónoma de Madrid), 28049 Madrid, Spain*

## ARTICLE INFO

### Article history:

Received 12 February 2015

Accepted 16 February 2015

Available online 15 May 2015

Communicated by Charles Fefferman

### Keywords:

Pointwise inequality

Laplace–Beltrami operators

Bochner subordination principle

Weyl’s law

Hausdorff–Bernstein–Widder theorem

Hele–Shaw flow

Quasi-geostrophic equation

## ABSTRACT

The fractional laplacian is an operator appearing in several evolution models where diffusion coming from a Lévy process is present but also in the analysis of fluid interphases. We provide an extension of a pointwise inequality that plays a rôle in their study. We begin recalling two scenarios where it has been used. After stating the results, for fractional Laplace–Beltrami and Dirichlet–Neumann operators, we provide a sketch of their proofs, unravelling the underlying principle to such inequalities.

© 2015 Elsevier Inc. All rights reserved.

## 1. Introduction

In this exposition we shall extend to a more general framework the following remarkable pointwise inequality

$$\Lambda^\alpha(\phi(f))(x) \leq \phi'(f(x)) \cdot \Lambda^\alpha f(x) \quad (1.1)$$

\* Corresponding author.

E-mail addresses: [antonio.cordoba@uam.es](mailto:antonio.cordoba@uam.es) (A. Córdoba), [angel.martinez@icmat.es](mailto:angel.martinez@icmat.es) (Á.D. Martínez).

valid for any convex function  $\phi \in C^1(\mathbb{R})$  and  $f$  in the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$ , where  $\Lambda = (-\Delta)^{\alpha/2}$  is defined as usual through the Fourier transform as

$$\widehat{\Lambda^\alpha \theta}(\xi) = |\xi|^\alpha \hat{\theta}(\xi)$$

The inequality holds also in the periodic setting and the proof provided in [5] follows directly from the representation

$$\Lambda^\alpha \theta(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{\theta(x) - \theta(y)}{|x - y|^{n+\alpha}} dy$$

where  $c_{n,\alpha} > 0$  together with the convexity hypothesis  $\phi(f(x)) - \phi(f(y)) \leq \phi'(f(x))(f(x) - f(y))$ . Despite its apparent simplicity its validity is quite surprising given the non-local character of the involved operators. It had also found several interesting applications to non-linear non-local evolution problems. Let us describe briefly two of them.

### 1.1. Transport equations

Suppose that we are confronted in  $\mathbb{R}^n$  or  $\mathbb{T}^n$  with the following initial value problem

$$\begin{cases} \theta_t(x, t) + u(x, t) \cdot \nabla_x \theta(x, t) = -\kappa \Lambda^\alpha \theta(x, t) \\ \theta(x, 0) = \theta_0(x) \end{cases}$$

where the velocity vector  $u$  is divergence-free and  $\kappa > 0$  is the viscosity coefficient. Then the pointwise inequality (1.1) is crucial to obtain the following maximum principle (see [6] for details):

$$\|\theta(\cdot, t)\|_p \leq \frac{\|\theta_0\|_p}{(1 + C\delta t \|\theta_0\|_p^{p\delta})^{1/p\delta}}$$

where  $\delta = \frac{\alpha}{2(p-1)}$ ,  $C = C(\kappa, \alpha, \|\theta_0\|_p) > 0$  and  $1 < p < \infty$ .

An important case of this is the surface quasi-geostrophic equation in which case the velocity field of the active scalar  $\theta$  is given in terms of the Riesz transforms as  $u = (-R_2\theta, R_1\theta)$ . In [6] the pointwise inequality was used to prove that the system above has solution valid in all time  $t > 0$  for any initial datum  $\theta_0$  in the Sobolev space  $H^1(\mathbb{R}^2)$ . It also appears, for example, in [4] where regularity results for critical diffusion ( $\alpha = 1$ ) are settled.

### 1.2. Interface evolution

In the theory of water waves or the Muskat and Hele-Shaw flows inside a porous media the inequality (1.1) had played a rôle in the study of the evolution of the free

Download English Version:

<https://daneshyari.com/en/article/4665277>

Download Persian Version:

<https://daneshyari.com/article/4665277>

[Daneshyari.com](https://daneshyari.com)