Advances in Mathematics 280 (2015) 79–85

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

A pointwise inequality for fractional laplacians

Antonio Córdoba, Ángel D. Martínez *

Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Departamento de Matemáticas (Universidad Autónoma de Madrid), 28049 Madrid, Spain

ARTICLE INFO

Article history:
Received 12 February 2015
Accepted 16 February 2015
Available online 15 May 2015
Communicated by Charles Fefferman

Keywords:
Pointwise inequality
Laplace–Beltrami operators
Bochner subordination principle
Weyl's law
Hausdorff–Bernstein–Widder theorem
Hele-Shaw flow
Quasi-geostrophic equation

ABSTRACT

The fractional laplacian is an operator appearing in several evolution models where diffusion coming from a Lévy process is present but also in the analysis of fluid interphases. We provide an extension of a pointwise inequality that plays a rôle in their study. We begin recalling two scenarios where it has been used. After stating the results, for fractional Laplace–Beltrami and Dirichlet–Neumann operators, we provide a sketch of their proofs, unravelling the underlying principle to such inequalities.

1. Introduction

In this exposition we shall extend to a more general framework the following remarkable pointwise inequality

$$\Lambda^\alpha(\phi(f))(x) \leq \phi'(f(x)) \cdot \Lambda^\alpha f(x) \tag{1.1}$$

* Corresponding author.
E-mail addresses: antonio.cordoba@uam.es (A. Córdoba), angel.martinez@icmat.es (Á.D. Martínez).

http://dx.doi.org/10.1016/j.aim.2015.02.018

valid for any convex function $\phi \in \mathcal{C}^1(\mathbb{R})$ and f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, where $\Lambda = (-\Delta)^{\alpha/2}$ is defined as usual through the Fourier transform as

$$\widehat{\Lambda^\alpha \theta}(\xi) = |\xi|^\alpha \hat{\theta}(\xi)$$

The inequality holds also in the periodic setting and the proof provided in [5] follows directly from the representation

$$\Lambda^\alpha \theta(x) = c_{n,\alpha} \int\limits_{\mathbb{R}^n} \frac{\theta(x) - \theta(y)}{|x-y|^{n+\alpha}} dy$$

where $c_{n,\alpha} > 0$ together with the convexity hypothesis $\phi(f(x)) - \phi(f(y)) \leq \phi'(f(x))(f(x) - f(y))$. Despite its apparent simplicity its validity is quite surprising given the non-local character of the involved operators. It had also found several interesting applications to non-lineal non-local evolution problems. Let us describe briefly two of them.

1.1. Transport equations

Suppose that we are confronted in $\mathbb{R}^n$ or $\mathbb{T}^n$ with the following initial value problem

$$\begin{cases} \theta_t(x,t) + u(x,t) \cdot \nabla_x \theta(x,t) = -\kappa \Lambda^\alpha \theta(x,t) \\ \theta(x,0) = \theta_0(x) \end{cases}$$

where the velocity vector u is divergence-free and $\kappa > 0$ is the viscosity coefficient. Then the pointwise inequality (1.1) is crucial to obtain the following maximum principle (see [6] for details):

$$\|\theta(\cdot,t)\|_p \leq \frac{\|\theta_0\|_p}{(1 + C\delta t \|\theta_0\|_p^{p\delta})^{1/p\delta}}$$

where $\delta = \frac{\alpha}{2(p-1)}$, $C = C(\kappa, \alpha, \|\theta_0\|_p) > 0$ and $1 < p < \infty$.

An important case of this is the surface quasi-geostrophic equation in which case the velocity field of the active scalar θ is given in terms of the Riesz transforms as $u = (-R_2\theta, R_1\theta)$. In [6] the pointwise inequality was used to prove that the system above has solution valid in all time $t > 0$ for any initial datum θ_0 in the Sobolev space $H^1(\mathbb{R}^2)$. It also appears, for example, in [4] where regularity results for critical diffusion $(\alpha = 1)$ are settled.

1.2. Interface evolution

In the theory of water waves or the Muskat and Hele-Shaw flows inside a porous media the inequality (1.1) had played a rôle in the study of the evolution of the free