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A Nadel vanishing theorem for metrics with minimal singularities on big line bundles

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ABSTRACT

The purpose of this paper is to establish a Nadel vanishing theorem for big line bundles with multiplier ideal sheaves of singular metrics admitting an analytic Zariski decomposition (such as, metrics with minimal singularities and Siu's metrics). For this purpose, we apply the theory of harmonic integrals and generalize Enoki's proof of Kollár's injectivity theorem. Moreover we investigate the asymptotic behavior of harmonic forms with respect to a family of regularized metrics.

1. Introduction

The Kodaira vanishing theorem plays an important role when we approach certain fundamental problems of algebraic geometry and the theory of several complex variables,

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for example, asymptotics of linear systems, extension problems of holomorphic sections, the minimal model program, and so on. By using multiplier ideal sheaves associated to singular metrics, this theorem is generalized to the Nadel vanishing theorem, which can be seen as an analytic analogue of the Kawamata–Viehweg vanishing theorem of algebraic geometry (see [13,25]).

In this paper, we study singular metrics admitting an analytic Zariski decomposition (such as, metrics with minimal singularities and Siu's metrics) and a Nadel vanishing theorem for them from the viewpoint of the theory of several complex variables and complex differential geometry.

Theorem 1.1 *(Nadel vanishing theorem, [20,21], cf. [2]). Let F be a big line bundle on a smooth projective variety X and h be a (singular) metric on F with (strictly) positive curvature. Then we have*

$$H^i(X, K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \textit{for any } i > 0.$$

Here $\mathcal{I}(h)$ denotes the multiplier ideal sheaf of the (singular) metric h and K_X denotes the canonical bundle of X.

We mainly handle metrics with minimal singularities $h_{\min}$ and Siu's metrics h_{Siu} (see Sections 2, 3 for the definition). These metrics satisfy many important properties (for example they admit an analytic Zariski decomposition), thus several authors study them (see [6,3]).

The main purpose of this paper is to establish a Nadel vanishing theorem for $h_{\min}$ and h_{Siu}. When we investigate the cohomology groups with coefficients in $K_X \otimes F \otimes \mathcal{I}(h_{\min})$ and $K_X \otimes F \otimes \mathcal{I}(h_{\mathrm{Siu}})$, we encounter the following difficulties:

(1) $h_{\min}$ and h_{Siu} may have non-algebraic (transcendental) singularities.
(2) $h_{\min}$ and h_{Siu} do not have strictly positive curvature except the trivial case.

The proof of Theorem 1.1 heavily depends on the assumption that the curvature of h is "strictly" positive. Under this assumption, we can construct solutions of the $\bar{\partial}$-equation with L^2-estimates, which implies Theorem 1.1 (see [2]). In fact the theorem fails even if the curvature of h is semi-positive. Nevertheless, we can expect that all higher cohomology groups with coefficients in $K_X \otimes F \otimes \mathcal{I}(h_{\min})$ and $K_X \otimes F \otimes \mathcal{I}(h_{\mathrm{Siu}})$ vanish from the special characteristics of $h_{\min}$ and h_{Siu}. This is because, for a big line bundle F we have already known

$$H^i(X, K_X \otimes F \otimes \mathcal{I}(\|F\|)) = 0 \quad \text{for any } i > 0,$$

where $\mathcal{I}(\|F\|)$ is the asymptotic multiplier ideal sheaf of F (see [6] for the precise definition). The multiplier ideal sheaves $\mathcal{I}(h_{\min})$ and $\mathcal{I}(h_{\mathrm{Siu}})$ can be seen as an analytic counterpart of $\mathcal{I}(\|F\|)$. The asymptotic multiplier ideal sheaf $\mathcal{I}(\|F\|)$ does not always