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Norms of inner derivations for multiplier algebras of C^*-algebras and group C^*-algebras, II ☆

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ABSTRACT

The derivation constant $K(A) \geq \frac{1}{2}$ has been extensively studied for *unital* non-commutative C^*-algebras. In this paper, we investigate properties of $K(M(A))$ where $M(A)$ is the multiplier algebra of a non-unital C^*-algebra A. A number of general results are obtained which are then applied to the group C^*-algebras $A = C^*(G_N)$ where G_N is the motion group $\mathbb{R}^N \rtimes SO(N)$. Utilizing the rich topological structure of the unitary dual $\widehat{G_N}$, it is shown that, for $N \geq 3$,

$$K(M(C^*(G_N))) = \frac{1}{2}\left\lceil \frac{N}{2} \right\rceil.$$

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1. Introduction

For a C^*-algebra A, an elementary application of the triangle inequality shows that

$$\|D(a,A)\| \leq 2d(a,Z(A))$$

for all $a \in A$, where $D(a,A)$ is the inner derivation generated by a and $d(a,Z(A))$ is the distance from a to $Z(A)$, the centre of A. This leads naturally to the definition of $K(A)$ as the smallest number in $[0,\infty]$ such that

$$K(A)\|D(a,A)\| \geq d(a,Z(A))$$

for all $a \in A$ [3,28]. If the elements a are restricted to be self-adjoint then the corresponding constant is denoted by $K_s(A)$. If $A = B(H)$ (or, more generally, a non-commutative von Neumann algebra on a Hilbert space $H \neq \mathbb{C}$) then $K(A) = \frac{1}{2}$ [37,38]. For unital non-commutative C^*-algebras, $K_s(A) = \frac{1}{2}\,\mathrm{Orc}(A)$ [35], where the *connecting order* $\mathrm{Orc}(A) \in \mathbb{N} \cup \{\infty\}$ is determined by a graph structure in the primitive ideal space $\mathrm{Prim}(A)$ (see Section 2), and for the constant $K(A)$ it has been shown that the only possible positive values less than or equal to $\frac{1}{2} + \frac{1}{\sqrt{3}}$ are

$$\frac{1}{2}, \quad \frac{1}{\sqrt{3}}, \quad 1, \quad \frac{3+8\sqrt{2}}{14}, \quad \frac{4}{\sqrt{15}}, \quad \frac{1}{2} + \frac{1}{\sqrt{3}}$$

[36,10,11]. These results use the fine structure of the topology on $\mathrm{Prim}(A)$ together with spectral constructions and the constrained optimization of the bounding radii of planar sets.

If A is a non-unital C^*-algebra then, as discussed in [7], the multiplier algebra $M(A)$ is the natural unitization to consider in the context of inner derivations. For example, it is well-known that if A is a primitive C^*-algebra then so is $M(A)$ (cf. [7, Example 5.5]) and so $K(M(A)) = \frac{1}{2}$ [37, Theorem 5]. In particular, $K(M(A)) = \frac{1}{2}$ for every simple C^*-algebra A.

In general, in order to apply to $M(A)$ the results for unital algebras, there is a prima facie requirement for more detailed information on $\mathrm{Prim}(M(A))$. However, this space is usually much larger and more complicated than the dense open subset $\mathrm{Prim}(A)$. This is illustrated by the complexity of the Stone–Čech compactification $\beta\mathbb{N}$ of the natural numbers $\mathbb{N}$ and also by the results in [13], which apply to the motion group C^*-algebras considered in this paper (see the remarks after Theorem 3.3). However, when A is σ-unital, the normality of the complete regularization of $\mathrm{Prim}(A)$ enables ideal structure in $M(A)$ to be linked to ideal structure in A without having full knowledge of $\mathrm{Prim}(M(A))$ (Proposition 2.1). It follows from this that, in several cases of interest, the value of $K(M(A))$ is determined by the ideal structure in A itself and hence by the topological properties of the T_0-space $\mathrm{Prim}(A)$ [20, 3.1]. This allows the possibility of computing $K(M(A))$ for $A = C^*(G)$ in cases where G is a locally compact group whose