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Random triangle removal ☆

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ABSTRACT

Starting from a complete graph on n vertices, repeatedly delete the edges of a uniformly chosen triangle. This stochastic process terminates once it arrives at a triangle-free graph, and the fundamental question is to estimate the final number of edges (equivalently, the time it takes the process to finish, or how many edge-disjoint triangles are packed via the random greedy algorithm). Bollobás and Erdős (1990) conjectured that the expected final number of edges has order $n^{3/2}$. An upper bound of $o(n^2)$ was shown by Spencer (1995) and independently by Rödl and Thoma (1996). Several bounds were given for variants and generalizations (e.g., Alon, Kim and Spencer (1997) and Wormald (1999)), while the best known upper bound for the original question of Bollobás and Erdős was $n^{7/4+o(1)}$ due to Grable (1997). No nontrivial lower bound was available.

Here we prove that with high probability the final number of edges in random triangle removal is equal to $n^{3/2+o(1)}$, thus confirming the $3/2$ exponent conjectured by Bollobás and Erdős and matching the predictions of Gordon, Kuperberg, Patashnik, and Spencer (1996). For the upper bound, for any fixed $\varepsilon > 0$ we construct a family of $\exp(O(1/\varepsilon))$ graphs by gluing $O(1/\varepsilon)$ triangles sequentially in a prescribed manner, and dynamically track the number of all homomorphisms from them, rooted at any two vertices, up to the point where

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$n^{3/2+\varepsilon}$ edges remain. A system of martingales establishes concentration for these random variables around their analogous means in a random graph with corresponding edge density, and a key role is played by the self-correcting nature of the process. The lower bound builds on the estimates at that very point to show that the process will typically terminate with at least $n^{3/2-o(1)}$ edges left.

1. Introduction

Consider the following well-known stochastic process for generating a triangle-free graph, and at the same time creating a partial Steiner triple system. Start from a complete graph on n vertices and proceed to repeatedly remove the edges of uniformly chosen triangles. That is, letting $G(0)$ denote the initial graph, $G(i+1)$ is obtained from $G(i)$ by selecting a triangle uniformly at random out of all triangles in $G(i)$ and deleting its 3 edges. The process terminates once no triangles remain, and the fundamental question is to estimate the stopping time

$$\tau_0 = \min\{i : G(i) \text{ is triangle-free}\}.$$

This is equivalent to estimating the number of edges in the final triangle-free graph, since $G(i)$ has precisely $\binom{n}{2} - 3i$ edges by definition. As the triangles removed are mutually edge-disjoint, this process is precisely the random greedy algorithm for triangle packing.

Bollobás and Erdős (1990) conjectured that the expected number of edges in $G(\tau_0)$ has order $n^{3/2}$ (see, e.g., [6,7]), with the motivation of determining the Ramsey number $R(3,t)$. Behind this conjecture was the intuition that the graph $G(i)$ should be similar to an Erdős–Rényi random graph with the same edge density. Indeed, in the latter random graph with n vertices and $\varepsilon n^{3/2}$ edges there are typically about $\frac{4}{3}\varepsilon^3 n^{3/2}$ triangles, thus, for small ε, deleting all of its triangles one by one would still retain all but a negligible fraction of the edges.

It was shown by Spencer [14] in 1995, and independently by Rödl and Thoma [13] in 1996, that the final number of edges is $o(n^2)$ with high probability (w.h.p.). In 1997, Grable [11] improved this to an upper bound of $n^{11/6+o(1)}$ w.h.p., and further described how similar arguments, using some more delicate calculations, should extend that result to $n^{7/4+o(1)}$. This remained the best upper bound prior to this work. No nontrivial lower bound was available. (See [10] for numerical simulations firmly supporting an answer of $n^{3/2+o(1)}$ to this problem.)

Of the various works studying generalizations and variants of the problem, we mention two here. In his paper from 1999, Wormald [15] demonstrated how the differential equation method can yield a nontrivial upper bound on the number of remaining edges during a greedy packing of hyperedges in k-uniform hypergraphs. For the special case $k = 3$, corresponding to triangle packing, this translated to a bound of $n^{2-\frac{1}{57}+o(1)}$. Also