Advances in Mathematics 284 (2015) 1–20

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The limit of the smallest singular value of random matrices with i.i.d. entries

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ARTICLE INFO

Article history:
Received 13 October 2014
Received in revised form 17 July 2015
Accepted 27 July 2015
Available online 5 August 2015
Communicated by D.W. Stroock

Keywords:
Random matrices
Smallest singular value

ABSTRACT

Let $\{a_{ij}\}$ $(1 \leq i, j < \infty)$ be i.i.d. real-valued random variables with zero mean and unit variance and let an integer sequence $(N_m)_{m=1}^{\infty}$ satisfy $m/N_m \longrightarrow z$ for some $z \in (0, 1)$. For each $m \in \mathbb{N}$ denote by A_m the $N_m \times m$ random matrix (a_{ij}) $(1 \leq i \leq N_m,\ 1 \leq j \leq m)$ and let $s_m(A_m)$ be its smallest singular value. We prove that the sequence $\left({N_m}^{-1/2} s_m(A_m)\right)_{m=1}^{\infty}$ converges to $1 - \sqrt{z}$ almost surely. Our result does not require boundedness of any moments of a_{ij}'s higher than the 2-nd and resolves a long standing question regarding the weakest moment assumptions on the distribution of the entries sufficient for the convergence to hold.

1. Introduction

For $N \geq m$ and an $N \times m$ real-valued matrix B, its *singular values* $s_1(B), s_2(B), \ldots, s_m(B)$ are the eigenvalues of the matrix $\sqrt{B^T B}$ arranged in non-increasing order, where multiplicities are counted. In particular, *the largest* and *the smallest* singular values are

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http://dx.doi.org/10.1016/j.aim.2015.07.020
0001-8708/

given by

$$s_1(B) = \sup_{y\in S^{m-1}} \|By\| = \|B\|; \quad s_m(B) = \inf_{y\in S^{m-1}} \|By\|.$$

In this paper, we establish convergence of the smallest singular values of a sequence random matrices with i.i.d. entries under minimal moment assumptions.

The extreme singular values of random matrices attract considerable attention of researchers both in *limiting* and *non-limiting* settings. We refer the reader to surveys and monographs [2,11,16,21] for extensive information on the spectral theory of random matrices. Here, we shall focus on the following specific question: for matrices with i.i.d. entries, what are the weakest possible assumptions on the entries which are sufficient for the smallest singular value to "concentrate"?

We note that a corresponding problem for the *largest* singular value (i.e. the operator norm) was essentially resolved in the i.i.d. case, where finiteness of the fourth moment of the entries turns out to be crucial both in limiting and non-limiting settings. We refer the reader to [24] and [3] for results on a.s. convergence of the largest singular value, and [7] for the non-limiting case (see also [17,9] for some negative results on concentration of the operator norm).

For the *smallest* singular value, its concentration properties are relatively well understood in the i.i.d. case provided that the fourth moment of the matrix entries is bounded. A classical theorem of Bai and Yin [4] (see also [2, Theorem 5.11]) states the following: given an array $\{a_{ij}\}$ $(1 \le i,j < \infty)$ of i.i.d. random variables such that $\mathbb{E}a_{ij} = 0$, $\mathbb{E}{a_{ij}}^2 = 1$ and $\mathbb{E}{a_{ij}}^4 < \infty$, and an integer sequence $(N_m)_{m=1}^{\infty}$ with $m/N_m \longrightarrow z$ for some $z \in (0,1)$, the $N_m \times m$ matrices $A_m = (a_{ij})$ $(1 \le i \le N_m,\ 1 \le j \le m)$ satisfy

$${N_m}^{-1/2} s_m(A_m) \longrightarrow 1 - \sqrt{z} \quad \text{almost surely.}$$

Further, it is proved in [13,14] that for square $m \times m$ matrices with i.i.d. centered entries with unit variance and a bounded fourth moment, one has $s_m(A) \approx m^{-1/2}$ with a large probability.

A natural question in connection with the mentioned results is *whether the assumption on the fourth moment is necessary for the least singular value to "concentrate"*; in particular, whether any assumptions on moments of a_{ij}'s higher than the 2-nd are required for the a.s. convergence in the Bai–Yin theorem. This question is discussed in [2] on p. 6. Solving the problem was a motivation for our work.

A considerable progress has been made recently in the direction of weakening the moment assumptions on matrix entries. For square matrices, given a sufficiently large m and an $m\times m$ matrix with i.i.d. entries with zero mean and unit variance, its smallest singular value is bounded from below by a constant (negative) power of m with probability close to one [19, Theorem 2.1] (see also [5, Theorem 4.1] for sparse matrices).

For *tall* rectangular matrices, Srivastava and Vershynin proved in [18] that for any $\varepsilon, \eta > 0$ and an $N \times m$ random matrix A with independent isotropic rows X_i such