Advances in Mathematics 284 (2015) 21–53

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The Tarski numbers of groups

Mikhail Ershov [a,1], Gili Golan [b,2], Mark Sapir [c,*,3]

[a] Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA
[b] Department of Mathematics, Bar-Ilan University, Ramat Gan 5290002, Israel
[c] Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

ARTICLE INFO

Article history:
Received 3 February 2014
Received in revised form 17 June 2015
Accepted 20 July 2015
Available online 10 August 2015
Communicated by the Managing Editors of AIM

MSC:
primary 43A07, 20F65
secondary 20E18, 20F05, 20F50

Keywords:
Tarski number
Paradoxical decomposition
Amenability
L^2-Betti number
Cost
Golod–Shafarevich groups

ABSTRACT

The Tarski number of a non-amenable group G is the minimal number of pieces in a paradoxical decomposition of G. In this paper we investigate how Tarski numbers may change under various group-theoretic operations. Using these estimates and known properties of Golod–Shafarevich groups, we show that the Tarski numbers of 2-generated non-amenable groups can be arbitrarily large. We also use the cost of group actions to show that there exist groups with Tarski numbers 5 and 6. These provide the first examples of non-amenable groups without free subgroups whose Tarski number has been computed precisely.

* Corresponding author.
E-mail addresses: ershov@virginia.edu (M. Ershov), gili.golan@math.biu.ac.il (G. Golan), m.sapir@vanderbilt.edu (M. Sapir).

[1] The work of the first author was partially supported by the NSF grant DMS-1201452 and the Sloan Research Fellowship grant BR 2011-105.
[2] The work of the second author was partially supported by BSF grant T-2012-238 and the Ministry of Science and Technology of Israel.
[3] The work of the third author was partially supported by NSF DMS-1318716, DMS-1261557 and BSF 2010295.

http://dx.doi.org/10.1016/j.aim.2015.07.010
0001-8708/

1. Introduction

Recall the definition of a *paradoxical decomposition* of a group.

Definition 1.1. A group G admits a *paradoxical decomposition* if there exist positive integers m and n, disjoint subsets $P_1, \ldots, P_m, Q_1, \ldots, Q_n$ of G and elements $g_1, \ldots, g_m$, $h_1, \ldots, h_n$ of G such that

$$G = \bigcup_{i=1}^{m} P_i g_i = \bigcup_{j=1}^{n} Q_j h_j. \tag{1.1}$$

It is well known [32] that G admits a paradoxical decomposition if and only if it is non-amenable. The minimal possible value of $m + n$ in a paradoxical decomposition of G is called the *Tarski number* of G and denoted by $\mathcal{T}(G)$.

The definition stated above (with the elements $g_1, \ldots, g_m, h_1, \ldots, h_n$ acting on the left) appears both in [32] and [28]. A slightly different definition of a paradoxical decomposition (see, for example, [3]) requires the sets $P_1, \ldots, P_m$, $Q_1, \ldots, Q_n$ to cover the entire group G and each of the unions $\bigcup_{i=1}^{m} P_i g_i$ and $\bigcup_{j=1}^{n} Q_j h_j$ to be disjoint. This alternative definition leads to the same notion of Tarski number: this follows from the proof of [27, Proposition 1.2] and Remark 2.2 below, but for completeness we will prove the equivalence of the two definitions of Tarski numbers in Appendix A.

It is clear that for any paradoxical decomposition we must have $m \geq 2$ and $n \geq 2$, so the minimal possible value of Tarski number is 4. By a theorem of Jónsson and Dekker (see, for example, [28, Theorem 5.8.38]), $\mathcal{T}(G) = 4$ if and only if G contains a non-Abelian free subgroup.

The problem of describing the set of Tarski numbers of groups has been formulated in [3], and the following results have been proved there:

Theorem 1.2. *(See [3].)*

(i) The Tarski number of any torsion group is at least 6.
(ii) The Tarski number of any non-cyclic free Burnside group of odd exponent ≥ 665 *is between* 6 *and* 14.

For quite some time it was unknown if the set of Tarski numbers is infinite. That question was asked by Ozawa [24] and answered in the positive by the third author. For every $m \geq 1$ let Amen_m (resp. Fin_m) be the class of all groups where all m-generated subgroups are amenable (resp. finite). For example, Amen_1 is the class of all groups and Fin_1 is the class of all torsion groups. Clearly $\mathrm{Fin}_m \subseteq \mathrm{Amen}_m$ for every m. Ozawa noticed [24] that all groups in Amen_m have Tarski number at least $m+3$, and the third author observed that Fin_m (for every m) contains non-amenable groups. This immediately follows from two results about Golod–Shafarevich groups proved in [5] and [7] (see Section 4.1 below). Thus there exist non-amenable groups with arbitrarily large Tarski numbers.