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Simple examples of affine manifolds with infinitely many exotic models

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ABSTRACT

We give a simple general method of constructing affine varieties with infinitely many exotic models. In particular we show that for every $d > 1$ there exists a Stein manifold of dimension d which has uncountably many different structures of affine variety.

1. Introduction

Given any smooth complex affine variety X, one can ask if there exist smooth affine varieties Y non-isomorphic to X but which are biholomorphic to X when equipped with their underlying structures of complex analytic manifold. When such exist, these varieties Y could be called exotic models of X.

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Examples of affine varieties with exotic models have been found in dimensions two and three (see [3,13,15]). Moreover, in [9] we showed that for every $n \geq 7$ there are n-dimensional rational affine manifolds with exotic models. The aim of this note is to give a simple general method of constructing such examples. In particular we show that examples of affine varieties with uncountably many exotic structures exist in any dimension $d > 1$ (for $d = 1$ it is easy to see that exotic structures do not exist). Here we modify our idea from [9] and we prove:

Theorem 1.1. *Let V be a non-rational smooth affine curve. Then:*

(i) The affine surface $Y := V \times \mathbb{C}$ has uncountably many different exotic models.

(ii) For every non-$\mathbb{C}$-uniruled smooth affine variety Z the variety $Y \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ of regular automorphisms of $V \times Z$ is at most countable, then the Stein manifold $Y \times Z$ has uncountably many different structures of affine variety.

Remark 1.2. If a variety Z is not uniruled, then the group $Aut(V \times Z)$ is automatically finite by [10].

As a consequence we have:

Corollary 1.3. *Let $\Gamma_1, \ldots, \Gamma_r$ be a finite collection of smooth affine non-rational curves ($r \geq 1$) and let $X = (\prod_{i=1}^{r} \Gamma_i) \times \mathbb{C}$. Then the Stein manifold X has uncountably many different structures of affine variety. In particular for every $d > 1$ there exists a Stein manifold of dimension d which has uncountably many different structures of affine variety.*

Moreover we obtain:

Theorem 1.4. *Let V be a smooth affine surface which has a smooth completion $\overline{V}$ with an effective canonical class (i.e., $H^0(\overline{V}, K_{\overline{V}}) \neq 0$). Then:*

(i) The affine fourfold $X := V \times \mathbb{C}^2$ has infinitely many different exotic models.

(ii) For every non-$\mathbb{C}$-uniruled smooth affine variety Z the variety $X \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ of regular automorphisms of $V \times Z$ is finite, then the Stein manifold $X \times Z$ has infinitely many different structures of affine variety.

Remark 1.5. In particular we can take as V (above) any generic surface $V \subset \mathbb{C}^3$ of degree $d \geq 4$. Moreover, for such V, if a variety Z is not uniruled, then the group $Aut(V \times Z)$ is automatically finite by [10].

Finally, we give the following geometric counterpart of Theorem 1.1:

Theorem 1.6. *Let $\overline{V}$ be a smooth projective curve of genus $g > 1$ and let $O \in \overline{V}$ be a sufficiently general point such that $Aut(\overline{V} \setminus \{O\})$ is a trivial group. Take $V = \overline{V} \setminus \{O\}$. Then*