Advances in Mathematics 287 (2016) 347–388

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Symmetries of statistics on lattice paths between two boundaries

Sergi Elizalde [a,*,1], Martin Rubey [b,2]

[a] *Department of Mathematics, Dartmouth College, Hanover, NH 03755, USA*
[b] *Institut für Diskrete Mathematik und Geometrie, TU Wien, Austria*

A R T I C L E I N F O

Article history:
Received 12 February 2015
Received in revised form 21 September 2015
Accepted 29 September 2015
Communicated by Erwin Lutwak

MSC:
primary 05A19
secondary 52C05, 05A15, 05B35, 05E05

Keywords:
Lattice path
Combinatorial statistic
Bijection
Triangulation
Semistandard Young tableau
Flagged tableau
Tutte polynomial
Matroid

A B S T R A C T

We prove that on the set of lattice paths with steps $N = (0, 1)$ and $E = (1, 0)$ that lie between two fixed boundaries T and B (which are themselves lattice paths), the statistics 'number of E steps shared with B' and 'number of E steps shared with T' have a symmetric joint distribution. To do so, we give an involution that switches these statistics, preserves additional parameters, and generalizes to paths that contain steps $S = (0, -1)$ at prescribed x-coordinates. We also show that a similar equidistribution result for path statistics follows from the fact that the Tutte polynomial of a matroid is independent of the order of its ground set. We extend the two theorems to k-tuples of paths between two boundaries, and we give some applications to Dyck paths, generalizing a result of Deutsch, to watermelon configurations, to pattern-avoiding permutations, and to the generalized Tamari lattice. Finally, we prove a conjecture of Nicolás about the distribution of degrees of k consecutive vertices in k-triangulations of a convex n-gon. To achieve this goal, we provide a new statistic-preserving bijection between certain k-tuples of non-crossing

* Corresponding author.
E-mail addresses: sergi.elizade@dartmouth.edu (S. Elizalde), martin.rubey@tuwien.ac.at (M. Rubey).

[1] The first author partially supported by grant DMS-1001046 from the NSF, by grant #280575 from the Simons Foundation, and by grant H98230-14-1-0125 from the NSA.
[2] The second author partially supported by EC's IHRP Programme, grant HPRN-CT-2001-00272, "Algebraic Combinatorics in Europe" and the European Research Council (ERC), Project number 306445.

http://dx.doi.org/10.1016/j.aim.2015.09.025

paths and k-flagged semistandard Young tableaux, which is based on local moves reminiscent of *jeu de taquin*.

1. Introduction[3]

In the first few sections of this paper we present two general theorems about lattice paths, together with several applications. Both theorems concern the set of lattice paths taking unit north and east steps, starting at the origin and ending at some prescribed point (x, y). Additionally, the paths are required to stay within a fixed region, whose boundaries are also lattice paths from the origin to (x, y).

Informally, the first theorem states that the joint distribution of the number of contacts with the top boundary and the number of contacts with the bottom boundary of the given set of lattice paths is symmetric. We provide a bijective proof of a generalization to paths that also contain south steps in Section 3. The second theorem states that the joint distribution of the number of top and the number of right contacts coincides with the joint distribution of the number of bottom and the number of left contacts. Its proof uses matroid theory, and it is presented in Section 4 and Appendix A. When specialized to a triangular region, both theorems reduce to a result of Emeric Deutsch [12,13] stating that, over the set of Dyck paths, the joint distribution of the number of returns and the height of the first peak is symmetric.

Some applications of our theorems are described in Section 6, including enumerative results on lattice paths with contacts in various regular regions, and a symmetry property on permutations with occurrences of a certain pattern at prescribed positions. We also establish a link between a map used in our proof and the covering relation in the generalized Tamari lattices recently introduced by François Bergeron [5].

Both theorems generalize naturally to families of non-intersecting paths, as shown in Section 5. Again, a particular case of interest is obtained by confining the paths to a triangular region, in which case the configurations are known in physics as 'watermelons'. To enumerate watermelons according to the number of contacts, we can use our result to reduce the problem to the enumeration of non-intersecting paths in a slightly modified region (see Section 6.5). This can then be tackled using the classical Lindström–Gessel–Viennot lemma, as shown by Christian Krattenthaler [28].

Section 7 contains our second main contribution, which is a new connection between two seemingly separate topics in algebraic combinatorics. We exhibit a natural weight-preserving bijection between k-flagged semistandard Young tableaux of shape λ, as appearing when studying Schubert polynomials [36], and families of k non-intersecting paths confined to a region of the same shape λ. We use this bijection to establish a

[3] For a slightly more elaborate introduction, we refer the reader to the arXiv preprint [17].