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A maximal function characterization for Hardy spaces associated to nonnegative self-adjoint operators satisfying Gaussian estimates

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ABSTRACT

Let L be a nonnegative, self-adjoint operator satisfying Gaussian estimates on $L^2(\mathbb{R}^n)$. In this article we give an atomic decomposition for the Hardy spaces $H^p_{L,\max}(\mathbb{R}^n)$ in terms of the nontangential maximal functions associated with the heat semigroup of L, and this leads eventually to characterizations of Hardy spaces associated to L, via atomic decomposition or the nontangential maximal functions. The proof is based on a modification of a technique due to A. Calderón [6].

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1. Introduction

The introduction and development of Hardy spaces on Euclidean spaces $\mathbb{R}^n$ in the 1960s played an important role in modern harmonic analysis and applications in partial differential equations. Let us recall the definition of the Hardy spaces (see [8,10,15,22,23,25,26]). Consider the Laplace operator $\Delta = -\sum_{i=1}^n \partial_{x_i}^2$ on the Euclidean spaces $\mathbb{R}^n$. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{R}^n)$ is defined as the space of tempered distribution $f \in \mathscr{S}'(\mathbb{R}^n)$ for which the area integral function of f satisfying

$$Sf(x) := \left(\int_0^\infty \int_{|y-x|<t} \left| t^2 \triangle e^{-t^2\triangle} f(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} \tag{1.1}$$

belongs to $L^p(\mathbb{R}^n)$. If this is the case, define

$$\|f\|_{H^p(\mathbb{R}^n)} := \|Sf\|_{L^p(\mathbb{R}^n)}. \tag{1.2}$$

When $p > 1$, $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. For $p \le 1$, the space $H^p(\mathbb{R}^n)$ involves many different characterizations. For example, if $f \in \mathscr{S}'(\mathbb{R}^n)$, then

$$\begin{aligned} f \in H^p(\mathbb{R}^n) &\overset{\text{(i)}}{\Longleftrightarrow} \sup_{t>0} \left| e^{-t^2\Delta} f(x) \right| \in L^p(\mathbb{R}^n) \\ &\overset{\text{(ii)}}{\Longleftrightarrow} \sup_{|y-x|<t} \left| e^{-t^2\Delta} f(y) \right| \in L^p(\mathbb{R}^n) \\ &\overset{\text{(iii)}}{\Longleftrightarrow} f \text{ has a } (p,q) \text{ atomic decomposition } f = \sum_{j=0}^\infty \lambda_j a_j \text{ with } \sum_{j=0}^\infty |\lambda_j|^p < \infty. \end{aligned} \tag{1.3}$$

Recall that a function a supported in ball B of $\mathbb{R}^n$ is called a (p,q)-atom, $0 < p \le 1 \le q \le \infty$, $p < q$, if $\|a\|_{L^q(B)} \le |B|^{\frac{1}{q}-\frac{1}{p}}$, and $\int_B x^\alpha a(x)dx = 0$, where α is a multi-index of order $|\alpha| \le \left[n(\frac{1}{p}-1)\right]$, the integer part of $n(\frac{1}{p}-1)$ (see [8,22,25]).

The theory of classical Hardy spaces has been very successful and fruitful in the past decades. However, there are important situations in which the standard theory of Hardy spaces is not applicable, including certain problems in the theory of partial differential equation which involve generalizing the Laplacian. There is a need to consider Hardy spaces that are adapted to a linear operator L, similarly to the way that the standard theory of Hardy spaces is adapted to the Laplacian. This topic has attracted a lot of attention in the last decades, and has been a very active research topic in harmonic analysis – see for example, [2–4,11,13,14,17–20,27].

In this article, we assume that L is a densely-defined operator on $L^2(\mathbb{R}^n)$ and satisfies the following properties: