Advances in Mathematics 287 (2016) 600–639

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Kernel estimates for nonautonomous Kolmogorov equations ☆

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A R T I C L E I N F O

Article history:
Received 24 September 2013
Received in revised form 9 July 2015
Accepted 18 September 2015
Communicated by Charles Fefferman

MSC:
35K10
35K08
37L40

Keywords:
Nonautonomous elliptic operators
Unbounded coefficients
Kernel estimates
Evolution systems of measures

A B S T R A C T

Using time dependent Lyapunov functions, we prove pointwise upper bounds for the heat kernels of some nonautonomous Kolmogorov operators with possibly unbounded drift and diffusion coefficients.

☆ MK and AR were supported by the DAAD-MIUR grant, Project Vigoni – ID 54710868; LL and AR were supported by the MIUR research project PRIN 2010MXMAJR.

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http://dx.doi.org/10.1016/j.aim.2015.09.029

1. Introduction

We establish kernel estimates for transition probabilities associated with nonautonomous evolution equations

$$\begin{cases} \partial_t u(t,x) = \mathscr{A}(t)u(t,x), & (t,x) \in (s,1] \times \mathbb{R}^d, \\ u(s,x) = f(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where the operators $\mathscr{A}(t)$ are defined on smooth functions φ by

$$(\mathscr{A}(t)\varphi)(x) = \sum_{ij=1}^{d} q_{ij}(t,x)D_{ij}\varphi(x) + \sum_{i=1}^{d} F_i(t,x)D_i\varphi(x)\,,$$

and $s \in [0,1)$. Parabolic equations like (1.1) appear naturally in connection with stochastic differential equations (cf. [26]), where it is natural to allow the coefficients to depend on time. Also in the study of Navier–Stokes flow past a rotating obstacle with Oseen condition, a suitable change of coordinates leads to a special class of nonautonomous Kolmogorov equations (cf. [12] and the references therein).

Throughout, we make the following assumptions on the coefficients.

Hypothesis 1.1. The coefficients q_{ij}, F_j $(i,j = 1,\ldots,d)$ are defined on $[0,1] \times \mathbb{R}^d$ and

(1) there exists an $\varsigma \in (0,1)$ such that $q_{ij}, F_j \in C^{\frac{\varsigma}{2},\varsigma}_{\mathrm{loc}}([0,1] \times \mathbb{R}^d)$ for all $i,j = 1,\ldots,d$. Moreover, $q_{ij} \in C^{0,1}((0,1) \times \mathbb{R}^d)$;
(2) the matrix $Q = (q_{ij})$ is symmetric and uniformly elliptic in the sense that there exists a number $\eta > 0$ such that

$$\sum_{i,j=1}^{d} q_{ij}(t,x)\xi_i\xi_j \geq \eta|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d,\ (t,x) \in [0,1] \times \mathbb{R}^d;$$

(3) there exist a nonnegative function $V \in C^2(\mathbb{R}^d)$ and a constant $M \geq 0$ such that $\lim_{|x|\to\infty} V(x) = \infty$ and we have $\mathscr{A}(t)V(x) \leq M$, as well as $\eta\Delta V(x) + F(t,x) \cdot \nabla V(x) \leq M$, for all $(t,x) \in [0,1] \times \mathbb{R}^d$.

Note that neither q_{ij} nor F_j $(i,j = 1,\ldots,d)$ are assumed to be bounded in $\mathbb{R}^d$.

Under Hypothesis 1.1, it was proved in [15] that equation (1.1) is well posed in the sense that, for every $f \in C_b(\mathbb{R}^d)$, there exists a unique function $u \in C_b([s,1] \times \mathbb{R}^d) \cap C^{1,2}((s,1] \times \mathbb{R}^d)$ such that (1.1) is satisfied. Moreover, there exists an evolution family $(G(t,s))_{t,s\in D} \subset \mathscr{L}(C_b(\mathbb{R}^d))$, where $D := \{(t,s) \in [0,1]^2 : t \geq s\}$, such that the unique solution u to (1.1) is given by $u = G(\cdot,s)f$. It turns out that each operator $G(t,s)$ is a contraction. We recall that an *evolution family* is a family $(G(t,s))_{(t,s)\in D}$ such that $G(t,t) = id_{C_b(\mathbb{R}^d)}$ and, for $r,s,t \in [0,1]$ with $r \leq s \leq t$, the *evolution law* $G(t,s)G(s,r) =$