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Approximation and convergence of the intrinsic volume ☆

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We introduce a modification of the classic notion of intrinsic volume using persistence moments of height functions. Evaluating the modified first intrinsic volume on digital approximations of a compact body with smoothly embedded boundary in $\mathbb{R}^n$, we prove convergence to the first intrinsic volume of the body as the resolution of the approximation improves. We have weaker results for the other modified intrinsic volumes, proving they converge to the corresponding intrinsic volumes of the n-dimensional unit ball.

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Contents

1. Introduction

Let $\mathbb{M}$ be a compact body in $\mathbb{R}^3$ whose boundary, $\partial\mathbb{M}$, is a smoothly embedded 2-manifold, and let t be a possibly small but positive real parameter. Letting $\#(\mathbb{M}\cap t\mathbb{Z}^3)$ be the number of points of the dilated integer grid in $\mathbb{M}$, it is well known that $t^3\#(\mathbb{M}\cap t\mathbb{Z}^3)$ converges to $\mathrm{Vol}(\mathbb{M})$ as t goes to zero. The central question of the classic lattice point theory, as founded by E. Landau and others in the first decades of the 20th century, is to estimate the *lattice discrepancy*, which is defined as $t^3\#(\mathbb{M}\cap t\mathbb{Z}^3) - \mathrm{Vol}(\mathbb{M})$; see the recent survey [19] for more details. Since the lattice discrepancy vanishes as t goes to zero, we may approximate $\mathbb{M}$ with $\#(\mathbb{M}\cap t\mathbb{Z}^3)$ cubes of edge length t whose centers are in $\mathbb{M}\cap t\mathbb{Z}^3$, such that the volume is preserved asymptotically, as t goes to zero. It would be nice to also preserve the other *intrinsic volumes* of $\mathbb{M}$, namely the surface area, the total mean curvature, and the total Gaussian curvature, by means of the above approximation with cubes. However, a straightforward construction only yields the right volume and Gaussian curvature, while the surface area and the mean curvature of the approximation can significantly differ from the values of $\mathbb{M}$ as the following example shows.

Motivating example. Let $\mathbb{M} = \mathbb{B}^3$ be the unit ball in $\mathbb{R}^3$. The *resolution t digital approximation* of $\mathbb{B}^3$, denoted as $\mathbb{B}^3_t$, is the union of axes-aligned cubes of edge length t whose centers are of the form (tx, ty, tz), with $(x, y, z) \in \mathbb{Z}^3$ and $t\sqrt{x^2 + y^2 + z^2} \leq 1$. There are $\#(\mathbb{B}^3 \cap t\mathbb{Z}^3) = \mathrm{Vol}(\mathbb{B}^3)/t^3 + o(1/t^3)$ such cubes, each with volume t^3. Hence,

$$\lim_{t\to 0} \mathrm{Vol}(\mathbb{B}^3_t) = \lim_{t\to 0} t^3\#(\mathbb{B}^3 \cap t\mathbb{Z}^3) = \mathrm{Vol}(\mathbb{B}^3). \tag{1}$$

As for the surface area, we note that if we look from either end of each of the three coordinate axes, we see every square face in the boundary of $\mathbb{B}^3_t$ exactly once. From each of the six directions, we see $\#(\mathbb{B}^2 \cap t\mathbb{Z}^2)$ faces, each of area t^2. As t goes to zero, the total area of these faces converges to the area of the unit disk, which implies

$$\lim_{t\to 0} \mathrm{Area}(\mathbb{B}^3_t) = 6 \lim_{t\to 0} t^2\#(\mathbb{B}^2 \cap t\mathbb{Z}^2) = 6\mathrm{Area}(\mathbb{B}^2), \tag{2}$$