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Test, multiplier and invariant ideals

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ABSTRACT

This paper gives an explicit formula for the multiplier ideals, and consequently for the log canonical thresholds, of any $\mathrm{GL}(V) \times \mathrm{GL}(W)$-invariant ideal in $S = \mathrm{Sym}(V \otimes W^*)$, where V and W are vector spaces over a field of characteristic 0. This characterization is done in terms of a polytope constructed from the set of Young diagrams corresponding to the Schur modules generating the ideal.

Our approach consists in computing the test ideals of some invariant ideals of S in positive characteristic: Namely, we compute the test ideals (and so the F-pure thresholds) of any sum of products of determinantal ideals. Not all the invariant ideals are as the latter (not even in characteristic 0), but they are up to integral closure, and this is enough to reach our goals.

The results concerning the test ideals are obtained as a consequence of general results holding true in a special situation. Within such framework fall determinantal objects of a generic matrix, as well as of a symmetric matrix and of a skew-symmetric one. Similar results are thus deduced for the $\mathrm{GL}(V)$-invariant ideals in $\mathrm{Sym}(\mathrm{Sym}^2 V)$ and in $\mathrm{Sym}(\bigwedge^2 V)$. (Monomial ideals also fall in this framework, thus we recover Howald's formula for their multiplier ideals and, more generally, Hara–Yoshida's formula for their test

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ideals.) In the proof, we introduce the notion of "floating test ideals", a property that in a sense is satisfied by ideals defining schemes with the nicest possible singularities. As will be shown, products of determinantal ideals, and by passing to characteristic 0 ideals generated by a single Schur module, have this property.

1. Introduction

Given an ideal $I \subseteq K[x_1, \dots, x_N]$, where K is a field of characteristic 0, its *multiplier ideals* $\mathscr{J}(\lambda \bullet I)$ (where $\lambda \in \mathbb{R}_{>0}$) are defined by meaning of a log-resolution. The *log canonical threshold* of I is the least λ such that $\mathscr{J}(\lambda \bullet I) \subsetneq K[x_1, \dots, x_N]$. In the words of Lazarsfeld [19], "the intuition is that these ideals will measure the singularities of functions $f \in I$, with 'nastier' singularities being reflected in 'deeper' multiplier ideals". In this paper, we give explicit formulas for the multiplier ideals (and therefore for the log canonical thresholds) of all the G-invariant ideals in a polynomial ring S, over a field of K characteristic 0, satisfying any of the following:

(i) $S = \mathrm{Sym}(V \otimes W^*)$, where V and W are finite K-vector spaces, $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$ and the action extends the diagonal one on $V \otimes W^*$ (Theorem 4.7).
(ii) $S = \mathrm{Sym}(\mathrm{Sym}^2 V)$, where V is a finite K-vector spaces, $G = \mathrm{GL}(V)$ and the action extends the natural one on $\mathrm{Sym}^2 V$ (Theorem 4.8).
(iii) $S = \mathrm{Sym}(\bigwedge^2 V)$, where V is a finite K-vector spaces, $G = \mathrm{GL}(V)$ and the action extends the natural one on $\bigwedge^2 V$ (Theorem 4.9).

The above results are obtained via reduction to characteristic $p > 0$: If $I \subseteq K[x_1, \dots, x_N]$, where K is a field of characteristic p, its *(generalized) test ideals* $\tau(\lambda \bullet I)$ (where $\lambda \in \mathbb{R}_{>0}$) are defined using notions from tight closure theory involving the Frobenius endomorphism. The connection between multiplier and test ideals is given by Hara and Yoshida [11], in a sense explaining why statements originally proved by using the theory of multiplier ideals often admit a proof also via the Hochster–Huneke theory of tight closure [14]: Roughly speaking, if $p \gg 0$, test ideals "coincide" with (the reduction mod p of) multiplier ideals. We give a general result for computing all test ideals of classes of ideals I satisfying certain conditions in a polynomial ring S over a field of characteristic $p > 0$ (Theorem 4.3). To give an idea, such conditions, quite combinatorial in nature, involve the existence of a polytope controlling the integral closure of the powers of I, and the existence of a pair consisting of a polynomial of S and a term ordering on S. This pair bares properties that depend on the coordinates of the real vector space in which the polytope lives (which correspond to suitable $\mathfrak{p} \in \mathrm{Spec}(S)$) and their weights (which are $\mathrm{ht}(\mathfrak{p})$) (see 4.1 for the precise definition). One can show that