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Determining plane curve singularities from its polars [☆]

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This paper addresses a very classical topic that goes back at least to Plücker: how to understand a plane curve singularity using its polar curves. Here, we explicitly construct the singular points of a plane curve singularity directly from the weighted cluster of base points of its polars. In particular, we determine the equisingularity class (or topological equivalence class) of a germ of plane curve from the equisingularity class of generic polars and combinatorial data about the non-singular points shared by them.

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Topological equivalence
Enriques diagram

1. Introduction

Polar germs are one of the main tools to analyze plane curve singularities, because they carry very deep analytical information on the singularity (see [21]). This holds still true for germs of hypersurfaces or even germs of analytic subsets of $\mathbb{C}^n$ (see for instance [29,30,21,20], or [13]). There have been lots of efforts in the literature with the aim of distinguishing which of this information is in fact purely topological. One of the first steps in solving this problem was settled more than thirty years ago by Teissier in [29]. There, he introduced the polar invariants, which in the planar case can be defined from the intersection multiplicity of the whole curve ξ with the branches of a generic polar, and he proved that they are topological invariants of ξ. This result has been generalized by Maugendre in [22] and by Michel in [24], where the role of polars is played by the Jacobian germs of planar morphisms and finite morphisms from normal surface singularities, respectively. The problem of relating a curve to its polars, and vice versa, is the motivation of lots of classical and recent works. Among these let us quote the works of Teissier [29,30], Merle [23], Kuo and Lu [17], Lê and Teissier [20], Eggers [10], Lê, Michel and Weber [18,19], Casas-Alvero [4,5], Gaffney [13], Delgado-de la Mata [8], García-Barroso [14], and García-Barroso and González-Pérez [15].

In this work we consider the classical topic of understanding a plane curve singularity ξ using its polar curves. The study of the contact between a reduced plane curve singularity and its polars goes back at least to Plücker, in 1837, in the framework of proving the global projective Plücker formulas [26]. This motivated later in 1875 the work of Smith [27], which is considered to be the first in giving local results on the contact between a germ of plane curve and its polars. The question addressed in this paper of determining a plane curve singularity from its polars implies solving two problems. The first one is to choose the right invariant, entirely computable from the polars, which determines the singular points of ξ (or its topological equivalence class), and this was solved by Casas-Alvero in [6, Theorem 8.6.4], in the way we will explain next. The second problem is to explicitly construct the singular points of ξ from this invariant, which is still open and is the scope of this work.

Regarding the first problem, the above mentioned polar invariants are computable from two polar curves taken in different directions (see Lemma 4.3), or equivalently from the weighted cluster of base points of the Jacobian system, and they could be a starting point. In fact, Merle showed in [23] that for an irreducible ξ the polar invariants and the multiplicity do determine its equisingularity class. However, this does not hold in general and there are examples of reducible non-equisingular curves with the same multiplicity and the same set of polar invariants (see [6, Example 6.11.7]). Another possibility could be to consider the topological class (or the singular points) of a generic polar, but it turns out that this analytic invariant carries not enough topological information about