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Maass–Jacobi Poincaré series and Mathieu Moonshine ☆

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ABSTRACT

Mathieu moonshine attaches a weak Jacobi form of weight zero and index one to each conjugacy class of the largest sporadic simple group of Mathieu. We introduce a modification of this assignment, whereby weak Jacobi forms are replaced by semi-holomorphic Maass–Jacobi forms of weight one and index two. We prove the convergence of some Maass–Jacobi Poincaré series of weight one, and then use these to characterize the semi-holomorphic Maass–Jacobi forms arising from the largest Mathieu group.

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Mathieu group
Mathieu moonshine

1. Introduction and statement of results

The Mathieu groups were discovered by Mathieu over 150 years ago [45,46]. Today, we recognize them as five of the 26 sporadic simple groups, serving as the exceptions to the general rule that most finite simple groups, namely the non-sporadic ones, are either cyclic of prime order, alternating of degree five or more, or finite of Lie type. This is the content of the classification of finite simple groups [1]. The largest Mathieu group, denoted M_{24}, may be characterized as the unique (up to isomorphism) proper subgroup of the alternating group of degree 24 that acts quintuply transitively on 24 points [19]. The stabilizer of a point in M_{24} is the sporadic Mathieu group M_{23}.

In 1988, Mukai established a surprising role for M_{23} in geometry by showing that it controls, in a certain sense, the finite automorphisms of certain complex manifolds of dimension two. More precisely, he proved [48] that any finite group of symplectic automorphisms of a complex K3 surface is isomorphic to a subgroup of M_{23} that has five orbits in the natural permutation representation on 24 points. Furthermore, any such subgroup may be realized via symplectic automorphisms of some complex K3 surface. For certain purposes, such as in mathematical physics, K3 surfaces serve as higher dimensional analogues of elliptic curves. For example, they are well-adapted to the constructions of string theory (see e.g. [2]). All complex K3 surfaces have the same diffeomorphism type, so we may consider one example, such as the Fermat quartic $\{X_1^4 + X_2^4 + X_3^4 + X_4^4 = 0\} \subset \mathbb{P}^3$, and regard the general K3 surface as a choice of complex structure on its underlying real manifold. Any K3 surface admits a non-vanishing holomorphic two-form, unique up to scale, and an automorphism that acts trivially on this two-form is called symplectic. We refer to [3,5] for more background on K3 surfaces.

In 2010, Eguchi, Ooguri, and Tachikawa made a stunning observation [26] that relates the largest Mathieu group M_{24} to K3 surfaces. To describe this, let $Z(\tau; z)$ be the unique weak Jacobi form of weight zero and index one satisfying $Z(\tau; 0) = 24$. It can be shown that Z may be written in the form (throughout $q := e^{2\pi i\tau}$)

$$Z(\tau; z) = 24\mathcal{A}(\tau; z)\frac{\theta_1(\tau; z)^2}{\eta(\tau)^3} + A(q)q^{-\frac{1}{8}}\frac{\theta_1(\tau; z)^2}{\eta(\tau)^3} \tag{1.1}$$

for some series $A(q) = \sum_{n\geq 0} A_n q^n \in \mathbb{Z}[[q]]$, where θ_1 is the usual Jacobi theta function

$$\theta_1(\tau; z) := i\sum_{n\in\mathbb{Z}}(-1)^n \zeta^{(n+\frac{1}{2})} q^{\frac{1}{2}(n+\frac{1}{2})^2},$$