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Self-similar subsets of the Cantor set

De-Jun Feng [a,*], Hui Rao [b], Yang Wang [c,d]

[a] Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
[b] Department of Mathematics and Statistics, Central China Normal University, Wuhan 430070, Hubei, Peoples's Republic of China
[c] Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong
[d] Department of Mathematics, Michigan State University, East Lansing, MI 48824, United States

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A B S T R A C T

In this paper, we study the following question raised by Mattila in 1998: what are the self-similar subsets of the middle-third Cantor set $\boldsymbol{C}$? We give criteria for a complete classification of all such subsets. We show that for any self-similar subset $\mathbf{F}$ of $\boldsymbol{C}$ containing more than one point, every linear generating IFS of $\mathbf{F}$ must consist of similitudes with contraction ratios $\pm 3^{-n}$, $n \in \mathbb{N}$. In particular, a simple criterion is formulated to characterize self-similar subsets of $\boldsymbol{C}$ with equal contraction ratio in modulus.

* Corresponding author.
E-mail addresses: djfeng@math.cuhk.edu.hk (D.-J. Feng), hrao@mail.ccnu.edu.cn (H. Rao), yangwang@ust.hk (Y. Wang).

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1. Introduction

Let $\boldsymbol{C}$ denote the standard middle-third Cantor set. The main goal of this paper is to answer the following open question raised by Mattila [2] in 1998: what are the self-similar subsets of $\boldsymbol{C}$?

Recall that a non-empty compact set $\mathbf{F} \subset \mathbb{R}$ is said to be *self-similar* if there exists a finite family $\Phi = \{\phi_i\}_{i=1}^k$ of contracting similarity maps on $\mathbb{R}$ such that

$$\mathbf{F} = \bigcup_{i=1}^{k} \phi_i(\mathbf{F}). \tag{1.1}$$

Such Φ is called a linear *iterated function system* (IFS) on $\mathbb{R}$. As proved by Hutchinson [5], for a given IFS Φ, there is a unique non-empty compact set $\mathbf{F}$ satisfying (1.1). To specify the relation between Φ and $\mathbf{F}$, we call Φ a *linear generating IFS* of $\mathbf{F}$, and $\mathbf{F}$ the *attractor* of Φ. Throughout this paper, we use $\mathbf{F}_\Phi$ to denote the attractor of a given linear IFS Φ. A self-similar set $\mathbf{F} = \mathbf{F}_\Phi$ is said to be *non-trivial* if it is not a singleton.

The middle-third Cantor set $\boldsymbol{C}$ is one of the most well known examples of self-similar sets. It has a generating IFS $\{x/3, (x+2)/3\}$.

The first result of this paper is the following theorem, which is our starting point for further investigations.

Theorem 1.1. *Assume that $\mathbf{F} \subseteq \boldsymbol{C}$ is a non-trivial self-similar set, generated by a linear IFS $\Phi = \{\phi_i\}_{i=1}^k$ on $\mathbb{R}$. Then for each $1 \le i \le k$, ϕ_i has contraction ratio $\pm 3^{-m_i}$, where $m_i \in \mathbb{N}$.*

The proof of the above theorem is based on a short geometric argument and a fundamental result of Salem and Zygmund on the sets of uniqueness in harmonic analysis.

It is easy to see that if a self-similar set $\mathbf{F}$ has a generating IFS $\Phi = \{\phi_i\}_{i=1}^k$ that is *derived* from the IFS $\Psi := \{x/3, (x+2)/3\}$, i.e., each map in Φ is a finite composition of maps in Ψ, then $\mathbf{F} \subseteq \boldsymbol{C}$. In light of Theorem 1.1, one may guess that each nontrivial self-similar subset of $\boldsymbol{C}$ has a linear generating IFS derived from Ψ. However, this is not true. The following counter example was constructed in [4].

Example 1.2. Let $\Phi = \{\frac{1}{9}x, \frac{1}{9}(x+2)\}$. Choose a sequence $(\epsilon_n)_{n=1}^\infty$ with $\epsilon_n \in \{0, 2\}$ so that $w = \sum_{n=1}^\infty \epsilon_n 3^{-2n+1}$ is an irrational number. Then by looking at the ternary expansion of the elements in $\mathbf{F}_\Phi + w := \{x + w : \ x \in \mathbf{F}_\Phi\}$, it is easy to see that $\mathbf{F}_\Phi + w \subset \boldsymbol{C}$. Observe that $\mathbf{F}_\Phi + w$ is a self-similar subset of $\boldsymbol{C}$ since it is the attractor of the IFS $\Phi' = \{\frac{1}{9}(x+8w), \frac{1}{9}(x+2+8w)\}$. However no generating IFS of $\mathbf{F}_{\Phi'}$ can be derived from the original IFS $\{\psi_0 = x/3, \psi_1 = (x+2)/3\}$, since $w = \min \mathbf{F}_{\Phi'}$ cannot be the fixed point of any map $\psi_{i_1 i_2 \ldots i_n}$ composed from ψ_0, ψ_1 due to the irrationality of w.

The above construction actually shows that $\boldsymbol{C}$ has uncountably many non-trivial self-similar subsets, and indicates the non-triviality of Mattila's question.