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Rigidity and non-rigidity results for conformal immersions

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ABSTRACT

In this paper we show a quantitative rigidity result for the minimizer of the Willmore functional among all projective planes in $\mathbb{R}^n$ with $n \geq 4$. We also construct an explicit counterexample to a corresponding rigidity result in codimension one, by showing that an Enneper surface might split-off during a blow-up process. For conformal immersions of spheres with large enough Willmore energies, we construct explicit counterexamples to a quantitative rigidity result and this complements the recently obtained rigidity results in [14].

1. Introduction

It is a classical result of Codazzi that two-dimensional surfaces in euclidean space which are totally umbilic are parts of a round sphere or a plane. This result was made

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quantitative in two papers of De Lellis and Müller [5,6], in which they showed that closed surfaces in $\mathbb{R}^3$ with small enough tracefree second fundamental form A^0 in L^2 (or equivalently with Willmore energy close to its absolute minimum value 4π among all closed surfaces) have to be $W^{2,2}$-close to a round sphere. Additionally, the conformal factor of the pull-back metric has to be L^∞-close to the one of the round spheres. These results relied on delicate estimates for conformal immersions in the Hardy space which were derived by Müller and Sverak [22]. One of the key features of the estimates of De Lellis and Müller is that the $W^{2,2}$-estimate for the difference of the immersion and a standard immersion of a round sphere, resp. the L^∞-estimate of the difference of the conformal factors, depends linearly on the L^2-norm of A^0.

Recently the authors were able to extend these results to surfaces in $\mathbb{R}^n$, see [16].

Recall that for a smooth immersion $f : \Sigma \to \mathbb{R}^n$ of a closed surface, we have by the Gauß equations and the Gauß–Bonnet theorem

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |A_f|^2 \,\mathrm{d}\mu_f + \pi\chi(\Sigma) = \frac{1}{2}\int_\Sigma |A_f^0|^2 \,\mathrm{d}\mu_f + 2\pi\chi(\Sigma),$$

where

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |H_f|^2 \,\mathrm{d}\mu_f$$

is the Willmore energy of f. Critical points of $\mathcal{W}$ are called Willmore surfaces.

In the case of spherical surfaces Σ and $n = 3$, Bryant [2] was able to classify all (smooth) critical points f_W of $\mathcal{W}$. More precisely, he showed that they are inversions of complete minimal surfaces with finite total curvature and embedded planer ends. Additionally he showed that the Willmore energy is quantized in the sense that

$$\mathcal{W}(f_W) = 4\pi m,$$

where m is the number of ends of the minimal surface associated to f_W and the values $m = 2, 3$ are not attained since there are no minimal surfaces with two or three ends satisfying the above conditions. A similar result was shown to be true for $n = 4$ by Montiel [21]. The result of Bryant was extended to possibly branched Willmore spheres with at most three branch points (including multiplicity) by the first author and Nguyen [15]. It was shown that the Willmore energy remains quantized under this assumption, a fact which is no longer true without the restriction on the number of branch points as was observed by Chen and Li [4] and Ndiaye and the second author [23]. Once singularities are allowed, the energy values 8π and 12π show up and they are realized by inversions of the catenoid, resp. the Enneper surface and the trinoid.

In a recent paper, the first author and Nguyen [14], were able to obtain quantitative rigidity results for immersions which are close in energy to the inverted catenoid or the inverted Enneper surface and which have at least a multiplicity two point resp.