Advances in Mathematics 281 (2015) 1242–1273

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

A duality approach to the symmetry of Bernstein–Sato polynomials of free divisors

Luis Narváez Macarro [1]

Departamento de Álgebra & Instituto de Matemáticas (IMUS), Facultad de Matemáticas, Universidad de Sevilla, Apdo. 1160, 41080 Sevilla, Spain

ARTICLE INFO

Article history:
Received 5 November 2014
Received in revised form 7 June 2015
Accepted 9 June 2015
Available online 19 June 2015
Communicated by Karen Smith

MSC:
14F10
32C38

Keywords:
Bernstein–Sato polynomials
Free divisors
Logarithmic differential operators
Spencer resolutions
Lie–Rinehart algebras
Logarithmic connections

ABSTRACT

In this paper we prove that the Bernstein–Sato polynomial of any free divisor for which the $\mathcal{D}[s]$-module $\mathcal{D}[s]h^s$ admits a Spencer logarithmic resolution satisfies the symmetry property $b(-s-2) = \pm b(s)$. This applies in particular to locally quasi-homogeneous free divisors (for instance, to free hyperplane arrangements), or more generally, to free divisors of linear Jacobian type. We also prove that the Bernstein–Sato polynomial of an integrable logarithmic connection $\mathcal{E}$ and of its dual $\mathcal{E}^*$ with respect to a free divisor of linear Jacobian type are related by the equality $b_{\mathcal{E}}(s) = \pm b_{\mathcal{E}^*}(-s-2)$. Our results are based on the behaviour of the modules $\mathcal{D}[s]h^s$ and $\mathcal{D}[s]\mathcal{E}[s]h^s$ under duality.

0. Introduction

In [16] Granger and Schulze proved that the Bernstein–Sato polynomial of any reductive prehomogeneous determinant or of any regular special linear free divisor satisfies the equality $b(-s-2) = \pm b(s)$. Their proof is based on Sato's fundamental theorem

E-mail address: narvaez@algebra.us.es.
[1] Partially supported by MTM2010-19298, P12-FQM-2696, MTM2013-46231-P and FEDER.

http://dx.doi.org/10.1016/j.aim.2015.06.012
0001-8708/

for irreducible reductive prehomogeneous spaces. This symmetry property has been also checked for many other examples of linear (see for instance [16] and [32]) and non-linear free divisors (e.g. quasi-homogeneous plane curves and the examples in [27]). In this paper we prove the above symmetry property for free divisors for which the $\mathcal{D}[s]$-module $\mathcal{D}[s]h^s$ admits a logarithmic Spencer resolution (see Theorem (4.1) for a precise statement). This hypothesis holds for any free divisor of linear Jacobian type, and so for any locally quasi-homogeneous free divisor (for instance, free hyperplane arrangements or discriminants of stable maps [22, Corollary 6.13] in Mather's "nice dimensions" [24]; "nice dimensions" are those dimensions of source and target manifolds for which stable proper mappings are dense in the proper mappings).

The main ingredient of the proof is the explicit description of the $\mathcal{D}[s]$-dual of $\mathcal{D}[s]h^s$ by means of the logarithmic duality formula in [7,8].

Let us mention that for any quasi-homogeneous germ $h : (\mathbb{C}^d, 0) \to (\mathbb{C}, 0)$ with isolated singularity, its reduced Bernstein–Sato polynomial $\widetilde{b}(s) = \frac{b(s)}{s+1}$ satisfies the equality $\widetilde{b}(s) = \pm\widetilde{b}(-s-d)$. This result and ours suggest that both are extremal cases of a whole family of "pure" cases where symmetry properties occur with other intermediate shiftings (see Question (6.2)). One can expect even that in the "non-pure" cases, the factors of the Bernstein–Sato polynomial which break the symmetry appear as minimal polynomials of the action of s on other $\mathcal{D}[s]$-modules attached to our singularity (see for instance the examples in [28, §3]), possibly related with the microlocal structure.

Let us now comment on the content of the paper.

In Section 1 we recall the different conditions and hypotheses on free divisors we will use throughout the paper. In Section 2 we recall the logarithmic Bernstein construction and we study the hypotheses we will need later to prove our main results. In Section 3 we apply the duality formula in [8] to describe the $\mathcal{D}[s]$-dual of $\mathcal{D}[s]h^{\varphi(s)}$, where φ is a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[s]$, under the hypotheses studied in Section 2. In Section 4 we prove the symmetry property $b(-s-2) = \pm b(s)$ under the above hypotheses. The idea of the proof is the following: once we know that the $\mathcal{D}[s]$-dual of $\mathcal{D}[s]h^s$ (resp. of $\mathcal{D}[s]h^{s+1}$) is concentrated in degree 0 and is isomorphic to $\mathcal{D}[s]h^{-s-1}$ (resp. to $\mathcal{D}[s]h^{-s-2}$), we can compute the $\mathcal{D}[s]$-dual of the exact sequence

$$0 \to \mathcal{D}[s]h^{s+1} \to \mathcal{D}[s]h^s \to \mathcal{Q} := (\mathcal{D}[s]h^s) \,/\, \left(\mathcal{D}[s]h^{s+1}\right) \to 0$$

and deduce that the $\mathcal{D}[s]$-dual of $\mathcal{Q}$ is concentrated in degree 1 and is isomorphic to $\mathcal{D}[s]h^{-s-2}/\mathcal{D}[s]h^{-s-1}$. From here the symmetry property comes up. At the end of the section we give some applications to the logarithmic comparison problem and a characterization of the logarithmic comparison theorem for Koszul free divisors. In Section 5 we generalize the above results to the case of integrable logarithmic connections with respect to free divisors of linear Jacobian type. In Section 6 we have included some open questions dealing with the relationship between the results of [16] and ours, and with the symmetry properties of (reduced) Bernstein–Sato polynomials in the non-free case. Finally, and for the ease of the reader, we have included an Appendix A with a