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# The planar Orlicz Minkowski problem in the $L^1$-sense

Sun Yijing [a,*], Long Yiming [b]

[a] *Department of Mathematics, University of Chinese Academy of Sciences, Beijing 100049, PR China*
[b] *Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, PR China*

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ABSTRACT

In this paper, we solve $L_p$ Minkowski problem for $L^1$ data and all $p < 0$, and Orlicz Minkowski problem with two nonlinear terms in $L^1$ sense. A byproduct is the Blaschke–Santaló inequality, which was previously established for only constant data, and now is shown to hold for $L^1$ data.
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## 1. Introduction

Given a closed convex hypersurface in the Euclidean space $\mathbb{R}^N$, the Gauss map defines a homeomorphism between the hypersurface and the unit sphere $\mathbb{S}^{N-1}$. The surface area measure, $S_K$, of a convex body $K$ (compact convex set with non-empty interior) is a Borel measure on the unit sphere $\mathbb{S}^{N-1}$, defined for each Borel $\omega \subset \mathbb{S}^{N-1}$ by

* Corresponding author.
*E-mail addresses:* yjsun@ucas.ac.cn (S. Yijing), longym@nankai.edu.cn (L. Yiming).

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$$S_K(\omega) = \int\limits_{x \in \gamma_K^{-1}(\omega)} d\mathcal{H}^{N-1}(x),$$

where $\gamma_K : \partial K \to \mathbb{S}^{N-1}$ is the Gauss map defined on the boundary of $K$, and $\mathcal{H}^{N-1}$ is $(N-1)$-dimensional Hausdorff measure. One of the cornerstones of the Brunn–Minkowski theory of convex bodies is the Minkowski problem: what are the necessary and sufficient conditions on a Borel measure on $\mathbb{S}^{N-1}$ so that it is the surface area measure of a convex body in $\mathbb{R}^N$. If the measure has a density $g$ with respect to the Lebesgue measure of the unit sphere $\mathbb{S}^{N-1}$, the Minkowski problem is equivalent to the study of solution to the following Monge–Ampère equation on the unit sphere

$$det(u_{i,j} + \delta_{i,j}u) = g \quad \text{on} \quad \mathbb{S}^{N-1}$$

where $(u_{i,j})$ is the Hessian matrix of $u$ with respect to an orthonormal frame. More than a century ago, Minkowski himself solved the classical Minkowski problem for the case where the given measure is discrete [39]. The complete solution to this problem for arbitrary measures was given by Alexandrov [2], Fenchel and Jessen [13] (see Schneider [42] for details). For the regularity of the classical Minkowski problem see Lewy [26], Nirenberg [40], Calabi [7], Cheng and Yau [10], Pogorelov [41] and Caffarelli et al. [6].

The celebrated $L_p$ Minkowski problem asks what are the necessary and sufficient conditions on a Borel measure on $\mathbb{S}^{N-1}$ to guarantee that it is the $L_p$ surface area measure of a convex body. This problem was introduced by Lutwak in [30,31], which is now one of the central problems in convex geometric analysis. The associated equation for the $L_p$ Minkowski problem is the following one

$$u^{1-p} det(u_{i,j} + \delta_{i,j}u) = g \quad \text{on} \quad \mathbb{S}^{N-1}$$

where $g : \mathbb{S}^{N-1} \to \mathbb{R}$ is called "data". Note that curvature flows are closely related to this problem, for example, homothetic solutions to isotropic flows (classified by Andrews [3]) are solutions of $L_p$ Minkowski problem. This $L_p$ Brunn–Minkowski theory evolved enormously over the past years, see e.g., Ai, Andrews, Böröczky, Chen, Chou, Wang, Dou, Zhu, Gage, Li, Guan, Lin, Ma, Hug, Ivaki, Jiang, Lutwak, Yang, Zhang, Oliker, Meyer, Werner, Schütt, Stancu, Umanskiy, G. Zhu, [1,3,5,8,11,12,14–16,22–25, 30–34,37,38,43–46,51] for details. Many challenges remain for $p < 1$ and, particularly, for negative $p$.

Note that the even $L_p$ Minkowski problem is specially referred to the $L_p$ Minkowski problem under an even data. The even for $p > 1$ was solved by Lutwak [30] under the assumption that $p \neq N$. The regularity of the solutions was established by Lutwak and Oliker [32]. In [33], Lutwak, Yang and Zhang showed that, for $p \neq N$, the $L_p$ Minkowski problem is equivalent to a volume-normalized $L_p$ Minkowski problem, and solved the even volume-normalized $L_p$ Minkowski problem for all $p > 1$. The even case $p = 0$ has been recently solved by Böröczky, Lutwak, Yang and Zhang [5].

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