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Semisimple Hopf actions on Weyl algebras

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ABSTRACT

We study actions of semisimple Hopf algebras H on Weyl algebras A over an algebraically closed field of characteristic zero. We show that the action of H on A must factor through a group action; in other words, if H acts inner faithfully on A, then H is cocommutative. The techniques used include reduction modulo a prime number and the study of semisimple cosemisimple Hopf actions on division algebras.

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1. Introduction

Let k be an algebraically closed field of characteristic zero and H a semisimple Hopf algebra over k. In [4, Theorem 1.3], two of the authors showed that any action of H on a commutative domain over k factors through a group action. The goal of this paper is to extend this result to Weyl algebras. Our main result states:

Theorem *(Theorem 4.1). Any semisimple Hopf action on the Weyl algebra $\mathbf{A}_n(k)$ factors through a group action.*

An equivalent formulation would be the following: if H acts *inner faithfully* on $\mathbf{A}_n(k)$, then H is cocommutative. By definition, inner faithfulness means that the action of H does not factor through a quotient Hopf algebra of smaller dimension.

Note that when the action of H preserves the standard filtration of $\mathbf{A}_n(k)$, Theorem 4.1 can be deduced from [4, Proposition 5.4], since the associated graded algebra $\mathrm{gr}(\mathbf{A}_n(k))$ is a commutative domain. Our main achievement in this paper is to eliminate this assumption.

We also obtain the result above for H finite dimensional, not necessarily semisimple, provided that the action of H gives rise to a Hopf–Galois extension, see Theorem 4.2.

The proof of Theorem 4.1 relies on reduction modulo a prime number, which allows us to reduce to the case where the algebra satisfies a polynomial identity (or is PI, for short). In this case, its quotient field is a division algebra with an action of H (Lemma 3.1). We then use the following result, interesting by itself:

Proposition *(Proposition 3.3(ii)). Let H be a semisimple cosemisimple Hopf algebra of dimension d over an algebraically closed field F (of any characteristic). Let D be a division algebra over F of degree m. If $d!$ is coprime to m, then any action of H on D factors through a group action.*

Using these methods, we will establish more general results on semisimple and non-semisimple Hopf actions on quantized algebras in future work. In particular, these methods will apply to module algebras B so that:

(†) B_p, the reduction of B modulo a prime number p, is PI and the PI-degree of B_p is a power of p, for $p \gg 0$.

Such algebras include universal enveloping algebras of finite dimensional Lie algebras and algebras of differential operators of smooth irreducible affine varieties. This prompts the following question, which is of independent interest in Ring Theory.

Question. Let B be a $\mathbb{Z}_+$-filtered algebra over k with $\mathrm{gr}(B)$ a finitely generated commutative domain. Does (†) hold for any large prime p?