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Metabolic modules of finite group algebras over finite fields of characteristic two

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ABSTRACT

Let G be a finite group and let F be a finite field of characteristic 2. We introduce *F-special subgroups* and *F-special elements* of G. In the case where F contains a pth primitive root of unity for each odd prime p dividing the order of G (e.g. it is the case once F is a splitting field for all subgroups of G), the F-special elements of G coincide with real elements of odd order. We prove that a symmetric FG-module V is metabolic if and only if the restriction V_D of V to every F-special subgroup D of G is metabolic, and also, if and only if the characteristic polynomial on V defined by every F-special element of G is a square of a polynomial over F. Some immediate applications to characters, self-dual codes and Witt groups are given.

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1. Introduction

Let G be a finite group and F be a finite field. Let V be a finite-dimensional FG-module which carries a G-invariant non-degenerate bilinear form. We call V a *symmetric* (*symplectic* respectively) FG-module if the form is symmetric (alternating respectively).

Let V be a symmetric or symplectic FG-module. A submodule W of V is said to be *isotropic* if $W \subseteq W^{\perp}$; while W is said to be *self-perpendicular* if $W^{\perp} = W$. Furthermore, we say that V is *metabolic* if it contains a self-perpendicular submodule. Note that metabolic symplectic FG-modules are also known as *hyperbolic* modules in the literature, see [2,3,11,14,15,19] for example.

The theory on metabolic modules is powerful and fruitful in many areas. The metabolic symplectic modules are quite useful for investigating correspondences of characters, extensions of characters, monomial characters and M-groups, for example, see [2,10,14–16,19,24,25]. The study of metabolic symmetric modules is used to determine the existence of self-dual codes and to characterize the automorphisms of self-dual codes, see [3,4,6,17,18]. Also, in order to study the Frobenius–Schur indicators of group representations, J.G. Thompson [23] defined the *Witt kernel* of a symmetric or symplectic FG-module V to be $W^{\perp}/W$ for a maximal isotropic submodule W of V, which inherits the form on V and is unique up to G-isometry (see [23, Lemma 2.1]). Obviously, V is metabolic if and only if its Witt kernel is trivial.

Dade [2, Theorem 3.2] proved the following theorem which reduces the problem of determining whether or not a symplectic module is metabolic to the "semisimple case".

Theorem 1.1 *(Dade). Suppose that F is a finite field of odd characteristic p, that G is a p-solvable group, that H is a subgroup of p-power index in G, and that V is a symplectic FG-module whose restriction V_H to a symplectic FH-module is metabolic. Then V is metabolic.*

Unfortunately, the theorem does not hold for characteristic $p = 2$, even if F is a splitting field for all subgroups of G. A counterexample can be constructed over $\mathbb{F}_4[S_3]$, where $\mathbb{F}_4$ denotes the field with four elements and S_3 denotes the symmetric group on three letters. Not surprisingly, "characteristic 2" causes great difficulties in many problems.

Using the above Dade's theorem, Loukaki [15, Theorem A] present an effective criterion for finite groups of odd order.

Theorem 1.2 *(Loukaki). Suppose that F is a finite field of odd characteristic p, that G is a finite group of odd order, and that V is a symplectic FG-module whose restriction V_C to a symplectic FC-module is metabolic for every cyclic subgroup C of G. Then V is metabolic.*