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Structure of symplectic invariant Lie subalgebras of symplectic derivation Lie algebras

Shigeyuki Morita [a], Takuya Sakasai [a,*], Masaaki Suzuki [b]

[a] Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan

[b] Department of Frontier Media Science, Meiji University, 4-21-1 Nakano, Nakano-ku, Tokyo, 164-8525, Japan

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A B S T R A C T

We study the structure of the symplectic invariant part $\mathfrak{h}_{g,1}^{\mathrm{Sp}}$ of the Lie algebra $\mathfrak{h}_{g,1}$ consisting of symplectic derivations of the free Lie algebra generated by the rational homology group of a closed oriented surface Σ_g of genus g.

First we describe the orthogonal direct sum decomposition of this space which is induced by the canonical metric on it and compute it explicitly up to degree 20. In this framework, we give a general constraint which is imposed on the Sp-invariant component of the bracket of two elements in $\mathfrak{h}_{g,1}$. Second we clarify the relations among $\mathfrak{h}_{g,1}$ and the other two related Lie algebras $\mathfrak{h}_{g,*}$ and $\mathfrak{h}_g$ which correspond to the cases of a closed surface Σ_g with and without base point $* \in \Sigma_g$. In particular, based on a theorem of Labute, we formulate a method of determining these differences and describe them explicitly up to degree 20. Third, by giving a general method of constructing elements of $\mathfrak{h}_{g,1}^{\mathrm{Sp}}$, we reveal a considerable difference between two particular submodules of it, one is the Sp-invariant part of a certain ideal $\mathfrak{j}_{g,1}$ and the other is that of the Johnson image.

Finally we combine these results to determine the structure of $\mathfrak{h}_{g,1}$ completely up to degree 6 including the unstable cases where the genus 1 case has an independent meaning.

* Corresponding author.
E-mail addresses: morita@ms.u-tokyo.ac.jp (S. Morita), sakasai@ms.u-tokyo.ac.jp (T. Sakasai), macky@fms.meiji.ac.jp (M. Suzuki).

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In particular, we see a glimpse of the Galois obstructions explicitly from our point of view.

1. Introduction and statements of the main results

Let $\Sigma_{g,1}$ be a compact oriented surface of genus $g \geq 1$ with one boundary component and we denote its first integral homology group $H_1(\Sigma_{g,1};\mathbb{Z})$ simply by H and let $H_\mathbb{Q} = H \otimes \mathbb{Q}$. We denote by $\mathcal{L}_{g,1}$ the free graded Lie algebra generated by $H_\mathbb{Q}$ and let $\mathfrak{h}_{g,1}$ be the graded Lie algebra consisting of *symplectic* derivations of $\mathcal{L}_{g,1}$. Let $\mathfrak{h}^+_{g,1}$ be the ideal consisting of derivations with *positive* degrees. This Lie algebra was introduced in the theory of Johnson homomorphisms (see [21]) and has been investigated extensively. We also consider closely related Lie algebras, denoted by $\mathfrak{h}_{g,*}$ and $\mathfrak{h}_g$ which correspond to the cases of a closed surface Σ_g with and without base point $* \in \Sigma_g$.

Let $\mathrm{Sp}(2g,\mathbb{Q})$ be the symplectic group which we sometimes denote simply by Sp. If we fix a symplectic basis of $H_\mathbb{Q}$, then it can be considered as the standard representation of $\mathrm{Sp}(2g,\mathbb{Q})$. Each piece $\mathfrak{h}_{g,1}(k)$, $\mathfrak{h}_{g,*}(k)$, $\mathfrak{h}_g(k)$, of the three graded Lie algebras, is naturally an Sp-module so that it has an irreducible decomposition. Let $\mathfrak{h}^{\mathrm{Sp}}_{g,1}$ denote the Lie subalgebra of $\mathfrak{h}_{g,1}$ consisting of Sp-invariant elements. We denote by $\mathfrak{h}_{g,1}(2k)^{\mathrm{Sp}}$ the degree $2k$ part of this Lie subalgebra. We use similar notations for the other two cases $\mathfrak{h}_{g,*}$ and $\mathfrak{h}_g$.

In [27] a canonical metric on $(H_\mathbb{Q}^{\otimes 2k})^{\mathrm{Sp}}$ is defined and its application to the tautological algebra of the moduli space of curves is given. It turns out that this metric can be described as a direct consequence of a result of Hanlon and Wales [13]. In Section 2, we first recall this metric and give a quick proof by quoting their result.

Now we can consider $\mathfrak{h}_{g,1}(2k)^{\mathrm{Sp}}$ as a subspace of $(H_\mathbb{Q}^{\otimes(2k+2)})^{\mathrm{Sp}}$ so that it has the induced metric. To formulate our results, we use the following terminology. A Young diagram λ is denoted by $[\lambda_1\lambda_2\cdots\lambda_h]$ $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h)$ and the number of boxes in λ, namely $\lambda_1 + \cdots + \lambda_h$, is denoted by $|\lambda|$. We also denote the number of rows of λ, namely h in the above notation, by $h(\lambda)$. For a given Young diagram λ as above, the symbol λ^δ denotes another Young diagram $[\lambda_1\lambda_1\cdots\lambda_{h(\lambda)}\lambda_{h(\lambda)}]$ which has *multiple double floors* while the symbol 2λ denotes $[2\lambda_1\cdots 2\lambda_{h(\lambda)}]$. Also $\mathfrak{S}_k$ denotes the symmetric group of order k.

Theorem 1.1. *With respect to the canonical metric on $\mathfrak{h}_{g,1}(2k)^{\mathrm{Sp}}$, there exists an orthogonal direct sum decomposition*

$$\mathfrak{h}_{g,1}(2k)^{\mathrm{Sp}} \cong \bigoplus_{|\lambda|=k+1,\ h(\lambda)\leq g} H_\lambda$$