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Unexpected distribution phenomenon resulting from Cantor series expansions

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A B S T R A C T

We explore in depth the number theoretic and statistical properties of certain sets of numbers arising from their Cantor series expansions. As a direct consequence of our main theorem we deduce numerous new results as well as strengthen the known ones.

1. Introduction

We will prove a general result that will have six seemingly unrelated classes of number theoretic applications. Unfortunately, it will take several pages to state this result. After this we describe the applications and then prove our theorem.

The Q-Cantor series expansion, first studied by G. Cantor in [10] is a natural generalization of the b-ary expansion. Let $\mathbb{N}_k := \mathbb{Z} \cap [k, \infty)$. If $Q \in \mathbb{N}_2^{\mathbb{N}}$, then we say that Q is a

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basic sequence. If $\lim_{n\to\infty} q_n = \infty$, then we say that Q is **infinite in limit**. Given a basic sequence $Q = \{q_n\}_{n=1}^{\infty}$, the **$Q$-Cantor series expansion** of a real x in $\mathbb{R}$ is the (unique) expansion of the form

$$x = E_0(x) + \sum_{n=1}^{\infty} \frac{E_n(x)}{q_1 q_2 \cdots q_n}, \tag{1.1}$$

where $E_0(x) = \lfloor x \rfloor$ and $E_n(x)$ is in $\{0, 1, \cdots, q_n - 1\}$ for $n \geq 1$ with $E_n(x) \neq q_n - 1$ infinitely often. We will write E_n in place of $E_n(x)$ when there is no room for confusion. Moreover, we will abbreviate (1.1) with the notation $x = E_0.E_1E_2E_3\cdots$ w.r.t. Q. Clearly, the b-ary expansion is a special case of (1.1) where $q_n = b$ for all n. If one thinks of a b-ary expansion as representing an outcome of repeatedly rolling a fair b-sided die, then a Q-Cantor series expansion may be thought of as representing an outcome of rolling a fair q_1 sided die, followed by a fair q_2 sided die and so on.

The study of normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions was started by P. Erdős, A. Rényi and P. Turán. This early work was done by P. Erdős and A. Rényi in [17] and [18] and by A. Rényi in [36–38] and by P. Turán in [42].

We recall the following standard definitions (see [23]). An **asymptotic distribution function** $f : [0, 1] \to [0, 1]$ is a non-decreasing function such that $f(0) = 0$ and $f(1) = 1$. For a sequence of real numbers $\omega = \{x_n\}$ with $x_n \in [0, 1)$ and an interval $I \subseteq [0, 1]$, define $A_n(I, \omega) := \#\{i \leq n : x_i \in I\}$. A sequence of real numbers $\omega = \{x_n\}$ has asymptotic distribution function f if

$$\lim_{n\to\infty} \frac{A_n([0, x), \omega)}{n} = f(x).$$

For the rest of this paper we will abbreviate asymptotic distribution function as adf. We say that a sequence ω is **uniformly distributed mod** 1 if ω has $f(x) = x$ as its adf. For the rest of this paper we will abbreviate uniformly distributed mod 1 as u.d. mod 1. Clearly, not all sequences have an adf. A sequence of real numbers $\omega = \{x_n\}$ has **upper asymptotic distribution function** $\overline{f}$ if

$$\overline{\lim_{n\to\infty}} \frac{A_n([0, x), \omega)}{n} = \overline{f}(x).$$

The sequence ω has **lower asymptotic distribution function** $\underline{f}$ if

$$\liminf_{n\to\infty} \frac{A_n([0, x), \omega)}{n} = \underline{f}(x).$$

Every sequence of real numbers ω has an upper and a lower adf. We note that the sequence ω has adf $f(x)$ if and only if $f = \overline{f} = \underline{f}$.

A great deal of information about the b-ary expansion of a real number x may be obtained by studying the distributional properties of the sequence $\mathcal{O}_b(x) := \{b^n x\}_{n=0}^{\infty}$.