Advances in Mathematics 274 (2015) 118–166

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Largest integral simplices with one interior integral point: Solution of Hensley's conjecture and related results

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A R T I C L E   I N F O

Article history:
Received 17 January 2014
Accepted 17 December 2014
Available online 28 January 2015
Communicated by Erwin Lutwak

MSC:
primary 52B20
secondary 14M25, 90C11

Keywords:
Barycentric coordinates
Simplex
Toric Fano variety
Volume
Integral polytope
Lattice-free polytope
Lattice diameter
Sylvester sequence

A B S T R A C T

For each dimension d, d-dimensional integral simplices with exactly one interior integral point have bounded volume. This was first shown by Hensley. Explicit volume bounds were determined by Hensley, Lagarias and Ziegler, Pikhurko, and Averkov. In this paper we determine the exact upper volume bound for such simplices and characterize the volume-maximizing simplices. We also determine the sharp upper bound on the coefficient of asymmetry of an integral polytope with a single interior integral point. This result confirms a conjecture of Hensley from 1983. Moreover, for an integral simplex with precisely one interior integral point, we give bounds on the volumes of its faces, the barycentric coordinates of the interior integral point and its number of integral points. Furthermore, we prove a bound on the lattice diameter of integral polytopes with a fixed number of interior integral points. The presented results have applications in toric geometry and in integer optimization.

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http://dx.doi.org/10.1016/j.aim.2014.12.035
0001-8708/

1. Introduction

1.1. Background information

The main objective of this manuscript is to provide sharp upper bounds on the size of integral polytopes with exactly one interior integral point. Note that standard notation and terminology is defined at the beginning of Section 2. For given $k \in \mathbb{N} \cup \{0\}$, we introduce the following two families of polytopes:

$$\mathcal{P}^d(k) := \left\{P \subseteq \mathbb{R}^d \,:\, P \text{ polytope},\ \dim(P) = d,\ \operatorname{vert}(P) \subseteq \mathbb{Z}^d,\ \left|\operatorname{int}(P) \cap \mathbb{Z}^d\right| = k\right\}$$

and

$$\mathcal{S}^d(k) := \{S \in \mathcal{P}^d(k) \,:\, S \text{ is a simplex}\}.$$

Given an integer $k \geq 1$ and a dimension $d \in \mathbb{N}$, the set $\mathcal{P}^d(k)$ is finite up to affine transformations which preserve the integer lattice $\mathbb{Z}^d$. In particular, $\mathcal{S}^d(k)$ is finite as well. A natural way to prove finiteness of $\mathcal{P}^d(k)$ and $\mathcal{S}^d(k)$ is to bound the volume of their elements from above in terms of k and d. This was first done by Hensley [18].

In order to obtain a volume bound for $\mathcal{S}^d(1)$, Hensley proved a lower bound on the minimal barycentric coordinate of the single interior integral point of $S \in \mathcal{S}^d(1)$. The minimal barycentric coordinate is in one-to-one correspondence to the so-called coefficient of asymmetry of S about its interior integral point. The coefficient of asymmetry is defined as follows: the intersection of each line through the interior integral point with S is divided into two parts by this point. For each line, consider the ratio between the lengths of these two parts. Then the coefficient of asymmetry is the maximum of these ratios. It is easy to observe that for $S \in \mathcal{S}^d(1)$, the coefficient of asymmetry about its interior integral point is equal to $\frac{1}{\beta} - 1$, where β is the minimal barycentric coordinate of this point; see also [33, (3)]. Hensley's results led him to conjecture the following: The simplex $\operatorname{conv}(\{o, s_1e_1, \dots, s_de_d\}) \in \mathcal{S}^d(1)$ has maximal coefficient of asymmetry among all elements of $\mathcal{S}^d(1)$, where $(s_i)_{i\in\mathbb{N}}$ denotes the *Sylvester sequence*, which is given by

$$\begin{aligned} s_1 &:= 2, \\ s_i &:= 1 + \prod_{j=1}^{i-1} s_j \quad \text{for } i \geq 2. \end{aligned}$$

1.2. Overview of new results and related open questions

In this work, we confirm the above conjecture of Hensley by proving sharp lower bounds on the barycentric coordinates. Another conjecture of Hensley was that the simplex