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Liouville theorems involving the fractional Laplacian on a half space

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ABSTRACT

Let $\mathbb{R}^n_+$ be the upper half Euclidean space and let α be any real number between 0 and 2. Consider the following Dirichlet problem involving the fractional Laplacian:

$$\begin{cases} (-\Delta)^{\alpha/2} u = u^p, & x \in \mathbb{R}^n_+, \\ u \equiv 0, & x \notin \mathbb{R}^n_+. \end{cases} \tag{1}$$

Instead of using the conventional extension method of Caffarelli and Silvestre [8], we employ a new and direct approach by studying an equivalent integral equation

$$u(x) = \int_{\mathbb{R}^n_+} G(x,y) u^p(y) dy. \tag{2}$$

Applying the *method of moving planes in integral forms*, we prove the non-existence of positive solutions in the critical

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and subcritical cases under no restrictions on the growth of the solutions.

1. Introduction

The fractional Laplacian in $\mathbb{R}^n$ is a nonlocal operator, taking the form

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha}PV\int_{\mathbb{R}^n}\frac{u(x)-u(z)}{|x-z|^{n+\alpha}}dz \tag{3}$$

where α is any real number between 0 and 2 and PV stands for the Cauchy principal value.

In recent years, there has been a great deal of interest in using the fractional Laplacian to model diverse physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars (see [4,9,17,25] and the references therein). It also has various applications in probability and finance [1,2,6]. In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy process [2]. We refer the readers to Di Nezza, Palatucci, and Valdinoci's survey paper [23] for a detailed exposition of the function spaces involved in the analysis of the operator and a long list of relevant references.

Let

$$\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \mid x_n > 0\}$$

be the upper half Euclidean space. In this paper, we establish Liouville type theorems, the non-existence of positive solutions, to the Dirichlet problem for elliptic semilinear equations

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = u^p, & x \in \mathbb{R}^n_+, \\ u(x) \equiv 0, & x \notin \mathbb{R}^n_+. \end{cases} \tag{4}$$

There are several distinctly different ways to define the fractional Laplacian in a domain $\Omega \subset \mathbb{R}^n$, which coincide when the domain is the entire Euclidean space, but can otherwise be quite different. In particular, Cabré and Tan [6] have analyzed a very similar problem, taking as the fractional Laplacian the operator with the same eigenfunctions as the regular Laplacian, by extending to one further dimension. Another way is to restrict the integration to the domain:

$$(-\Delta)^{\alpha/2}_{\Omega}u(x) = C_{n,\alpha}PV\int_{\Omega}\frac{u(x)-u(z)}{|x-z|^{n+\alpha}}dz,$$

known as the regional fractional Laplacian [20].