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The Isbell monad ☆

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ABSTRACT

In 1966 [7], John Isbell introduced a construction on categories which he termed the "couple category" but which has since come to be known as the *Isbell envelope*. The Isbell envelope, which combines the ideas of contravariant and covariant presheaves, has found applications in category theory, logic, and differential geometry. We clarify its meaning by exhibiting the assignation sending a locally small category to its Isbell envelope as the action on objects of a pseudomonad on the 2-category of locally small categories; this is the *Isbell monad* of the title. We characterise the pseudoalgebras of the Isbell monad as categories equipped with a *cylinder factorisation system*; this notion, which appears to be new, is an extension of Freyd and Kelly's notion of factorisation system [5] from orthogonal classes of arrows to orthogonal classes of cocones and cones.

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1. Introduction

One of the most fundamental constructions in category theory is that which assigns to a small category $\mathcal{C}$ the Yoneda embedding $Y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ into its category of

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presheaves. As is well known, this embedding has the effect of exhibiting $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ as a *free cocompletion* of $\mathcal{C}$: the value at $\mathcal{C}$ of a left biadjoint

$$\mathbf{COCTS} \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} \mathbf{CAT} \tag{1.1}$$

to the forgetful 2-functor from small-cocomplete categories and cocontinuous functors to locally small ones. At a $\mathcal{C}$ which is not necessarily small, this left biadjoint still exists, but now has its unit $Y: \mathcal{C} \to \mathcal{P}\mathcal{C}$ given by the Yoneda embedding into the subcategory $\mathcal{P}\mathcal{C} \subset [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ of *small* presheaves: those which can be expressed as small colimits of representables. Composing the two biadjoints in (1.1) exhibits the process of free cocompletion as the functor part of a pseudomonad $\mathcal{P}$ on $\mathbf{CAT}$, and it turns out that the $\mathcal{P}$-pseudoalgebras and algebra pseudomorphisms are once again the small-cocomplete categories and cocontinuous functors between them; which is to say that the biadjunction (1.1) is *pseudomonadic* [12].

Dually, we speak of *free completions* of categories, meaning the values of a left biadjoint to the forgetful 2-functor $\mathbf{CTS} \to \mathbf{CAT}$ from complete categories to locally small ones. The free completion of a small $\mathcal{C}$ is witnessed by the dual Yoneda embedding $Y: \mathcal{C} \to [\mathcal{C}, \mathbf{Set}]^{\mathrm{op}}$, while the general completion $Y: \mathcal{C} \to \mathcal{P}^{\dagger}\mathcal{C}$ is constructed as $\mathcal{P}^{\dagger}\mathcal{C} = \mathcal{P}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \subset [\mathcal{C}, \mathbf{Set}]^{\mathrm{op}}$. As before, the biadjunction $\mathbf{CTS} \leftrightarrows \mathbf{CAT}$ induced by free completion is pseudomonadic, so that, as before, complete categories and continuous functors between them may be identified with $\mathcal{P}^{\dagger}$-pseudoalgebras and their pseudomorphisms.

In [7, §1.1], Isbell describes a construction that, in some sense, combines the processes of free completion and cocompletion; while Isbell calls this construction the "couple category", we follow Lawvere in terming it the *Isbell envelope*. Given a locally small category $\mathcal{C}$, the objects of its Isbell envelope $\mathcal{I}\mathcal{C}$ are triples (X^+, X^-, ξ^X) where $X^+ \in \mathcal{P}\mathcal{C}$ and $X^- \in \mathcal{P}^{\dagger}\mathcal{C}$ and $\xi^X_{ab}: X^-(b) \times X^+(a) \to \mathcal{C}(a, b)$ is a family of functions, natural in a and b; while morphisms $(X^+, X^-, \xi^X) \to (Y^+, Y^-, \xi^Y)$ in $\mathcal{I}\mathcal{C}$ are pairs (f^+, f^-), where $f^+: X^+ \to Y^+$ in $\mathcal{P}\mathcal{C}$ and $f^-: X^- \to Y^-$ in $\mathcal{P}^{\dagger}\mathcal{C}$ are such that each square

$$\begin{array}{ccc} Y^-(b) \times X^+(a) & \xrightarrow{1 \times f^+} & Y^-(b) \times Y^+(a) \\ \downarrow{\scriptstyle f^- \times 1} & & \downarrow{\scriptstyle \xi^Y} \\ X^-(b) \times X^+(a) & \xrightarrow{\xi^X} & \mathcal{C}(a, b) \end{array} \tag{1.2}$$

commutes in $\mathbf{Set}$. There is a Yoneda embedding $Y: \mathcal{C} \to \mathcal{I}\mathcal{C}$ into the Isbell envelope, whose value at an object c is given by:

$$\big(\mathcal{C}(-, c) \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}],\ \mathcal{C}(c, -) \in [\mathcal{C}, \mathbf{Set}],\ \big(\mathcal{C}(c, b) \times \mathcal{C}(a, c) \xrightarrow{\circ} \mathcal{C}(a, b)\big)_{a,b}\big),$$

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