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Bimodules in crossed products of von Neumann algebras

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In this paper, we study bimodules over a von Neumann algebra M in the context of an inclusion $M \subseteq M \rtimes_\alpha G$, where G is a discrete group acting on a factor M by outer $*$-automorphisms. We characterize the M-bimodules $X \subseteq M \rtimes_\alpha G$ that are closed in the Bures topology in terms of the subsets of G. We show that this characterization also holds for w^*-closed bimodules when G has the approximation property (AP), a class of groups that includes all amenable and weakly amenable ones. As an application, we prove a version of Mercer's extension theorem for certain w^*-continuous surjective isometric maps on X.

1. Introduction

The starting point for this paper is a theorem due to H. Choda [6] which describes the von Neumann algebras which lie between a factor M and its crossed product $M \rtimes_\alpha G$

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by a discrete group G acting on M by outer $*$-automorphisms. Choda proved that an intermediate von Neumann algebra N for which there exists a normal conditional expectation $E : M \rtimes_\alpha G \to N$ must have the form $M \rtimes_\alpha H$ for a subgroup H of G. When M is type II_1, the existence of E is immediate and so the intermediate von Neumann algebras are precisely of the form $M \rtimes_\alpha H$ for subgroups H of G. The result for other types of factors was improved in [14], where it was shown that such conditional expectations always exist, at least when M has separable predual. It is natural to ask whether the w^*-closed M-bimodules can be characterized in a similar way. Each subset S of G gives rise to a w^*-closed M-bimodule as $X_S = \overline{\mathrm{span}}^{w^*}\{mg : m \in M,\ g \in S\}$, so the question is whether all w^*-closed M-bimodules occur in this way. The first part of the paper is devoted to this problem, and we are able to provide a complete answer when we work with a different topology introduced by Bures in [4]. Our techniques are largely based on Fourier series in the crossed product and, as noted by Mercer [17], the Bures topology is the correct one for understanding the convergence of such series. We establish a correspondence between the Bures closed M-bimodules and the subsets of G, and recapture the results on intermediate von Neumann algebras above without separability restrictions. The Bures closed M-bimodules are all w^*-closed, but the reverse statement is open. However, for discrete groups with the approximation property (AP), the two types of bimodules coincide. The class of groups with the AP includes all amenable and weakly amenable discrete groups, and is closed under semidirect products.

Bimodules over subalgebras of von Neumann algebras have been studied previously in various contexts. For example, the case of an inclusion $A \subseteq M$ for a Cartan subalgebra A was examined in [19], but unfortunately the arguments for the Spectral Theorem for Bimodules contained a gap. This was known informally for several years prior to the publication of [1] which identified a problem with the proof. In the meantime, a proof for the special case when the containing von Neumann algebra M is amenable was given in [11]. A shorter proof of the Spectral Theorem for Bimodules in full generality was attempted in [18], but this regrettably also contained an error. A slightly different formulation of the result was found recently in [5] which characterized the Bures closed A-bimodule algebras in M rather than the w^*-closed ones, but the status of the original formulation of the Spectral Theorem for Bimodules remains uncertain. There are certainly w^*-closed subspaces that are not Bures closed, suggesting that the same phenomenon might occur for bimodules, but no such examples are known. We also note that bimodules have been studied in the related context of tensor products of von Neumann algebras by Kraus [16]. If N is a factor satisfying a technical condition called the *weak* operator approximation property* and R is a von Neumann algebra, then he was able to characterize the σ-weakly closed N-bimodules of $N \overline{\otimes} R$ as those subspaces of the form $N \overline{\otimes} T$ where T is a σ-weakly closed subspace of R.

If X is a w^*-closed unital subspace of a von Neumann algebra which it generates, then we may consider the inclusion $X \subseteq W^*(X)$. An old question, dating back at least to early work of Arveson [2,3], asks whether w^*-continuous unital surjective isometries on X extend to $*$-automorphisms of $W^*(X)$. This has been studied in [9,5], the latter