Advances in Mathematics 274 (2015) 562–605

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

What do homotopy algebras form?

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ARTICLE INFO

Article history:
Received 30 July 2014
Accepted 13 January 2015
Available online 11 February 2015
Communicated by Michel Van den Bergh

Keywords:
Homotopy algebras
Operads
Higher categories

ABSTRACT

In paper [4], we constructed a symmetric monoidal category $\mathfrak{S}\mathsf{Lie}_\infty^{\mathrm{MC}}$ whose objects are shifted (and filtered) L_∞-algebras. Here, we fix a cooperad $\mathcal{C}$ and show that algebras over the operad $\mathrm{Cobar}(\mathcal{C})$ naturally form a category enriched over $\mathfrak{S}\mathsf{Lie}_\infty^{\mathrm{MC}}$. Following [4], we "integrate" this $\mathfrak{S}\mathsf{Lie}_\infty^{\mathrm{MC}}$-enriched category to a simplicial category $\mathsf{HoAlg}_{\mathcal{C}}^{\Delta}$ whose mapping spaces are Kan complexes. The simplicial category $\mathsf{HoAlg}_{\mathcal{C}}^{\Delta}$ gives us a particularly nice model of an $(\infty, 1)$-category of $\mathrm{Cobar}(\mathcal{C})$-algebras. We show that the homotopy category of $\mathsf{HoAlg}_{\mathcal{C}}^{\Delta}$ is the localization of the category of $\mathrm{Cobar}(\mathcal{C})$-algebras and ∞-morphisms with respect to ∞-quasi-isomorphisms. Finally, we show that the Homotopy Transfer Theorem is a simple consequence of the Goldman–Millson theorem.

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http://dx.doi.org/10.1016/j.aim.2015.01.014
0001-8708/

1. Introduction

This work is motivated by Dmitry Tamarkin's answer [24] to Vladimir Drinfeld's question "What do dg categories form?" [10], and by papers [1,8,13,15,18]. Here, we give an answer to the question "What do homotopy algebras form?"

Homotopy algebras and their generalizations appear in constructions of string topology, in rational homotopy theory, symplectic topology, deformation quantization, and quantum field theory. For a gentle introduction to this topic, we refer the reader to paper [25]. For a more detailed exposition, a standard reference is book [19] by Jean-Louis Loday and Bruno Vallette.

Despite an important role of homotopy algebras, it is not clear what higher categorical structure stands behind the homotopy theory of homotopy algebras. A possible way to answer this question is to use closed model categories and this approach is undertaken in paper [26] by Bruno Vallette. Here, we suggest a different approach which is based on the use of the convolution L_∞-algebra and the Getzler–Hinich construction [13,15].

By a homotopy algebra structure on a cochain complex of $\Bbbk$-vector[1] spaces A we mean a Cobar$(\mathcal{C})$-algebra structure on A, where $\mathcal{C}$ is a differential graded (dg) cooperad satisfying some technical conditions. In this paper, we fix such a cooperad $\mathcal{C}$ and show that Cobar$(\mathcal{C})$-algebras form a $\mathfrak{S}\mathsf{Lie}_\infty^{\mathrm{MC}}$-enriched category $\mathsf{HoAlg}_{\mathcal{C}}$, where $\mathfrak{S}\mathsf{Lie}_\infty^{\mathrm{MC}}$ is the enhancement of the symmetric monoidal category of shifted L_∞-algebras introduced in[2] [4]. Then, using a generalization of the Getzler–Hinich construction [13,15], we show that the $\mathfrak{S}\mathsf{Lie}_\infty^{\mathrm{MC}}$-category of Cobar$(\mathcal{C})$-algebras can be "integrated" to the category $\mathsf{HoAlg}_{\mathcal{C}}^{\Delta}$ enriched over ∞-groupoids (a.k.a. Kan complexes).

Our approach is conceptually similar to a standard construction of the simplicial category of chain complexes, i.e., "trivial" homotopy algebras. There, instead of constructing the simplicial mapping spaces directly, one exploits the simple fact that chain complexes are naturally enriched over chain complexes, i.e., abelian L_∞-algebras. The simplicial category is then constructed by first truncating the mapping complexes and applying the Dold–Kan functor. It has been noted, for example by E. Getzler in [13], that the integration of an L_∞-algebra is, up to homotopy, a non-abelian analogue of the Dold–Kan functor. Hence, to construct the simplicial category $\mathsf{HoAlg}_{\mathcal{C}}^{\Delta}$ of non-trivial homotopy algebras, we proceed via analogy by first considering a category of homotopy algebras enriched over L_∞-algebras, i.e., "non-abelian complexes".

Furthermore, we prove that the homotopy category of $\mathsf{HoAlg}_{\mathcal{C}}^{\Delta}$ *is* the localization of the category of Cobar$(\mathcal{C})$-algebras with respect to ∞-quasi-isomorphisms, and this is indeed the correct homotopy category of homotopy algebras (see for example [19, Thm. 11.4.12]). Thus the simplicial category $\mathsf{HoAlg}_{\mathcal{C}}^{\Delta}$ is the sought higher categorical structure which stands behind the homotopy category of homotopy algebras.

[1] In this paper, we assume that $\mathrm{char}(\Bbbk) = 0$.
[2] See also Section 3.1 of this paper.