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Elliptic and parabolic equations with measurable coefficients in weighted Sobolev spaces

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ABSTRACT

We consider both divergence and non-divergence parabolic equations on a half space in weighted Sobolev spaces. All the leading coefficients are assumed to be only measurable in the time and one spatial variable except one coefficient, which is assumed to be only measurable either in the time or the spatial variable. As functions of the other variables the coefficients have small bounded mean oscillation (BMO) semi-norms. The lower-order coefficients are allowed to blow up near the boundary with a certain optimal growth condition. As a corollary, we also obtain the corresponding results for elliptic equations.

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1. Introduction

In this paper we study parabolic equations in non-divergence form and divergence form:

$$
\begin{aligned}
&-u_t + a^{ij} D_{ij} u + b^i D_i u + cu - \lambda u = f, \\
&-u_t + D_i\big(a^{ij} D_j u + b^i u\big) + \hat{b}^i D_i u + cu - \lambda u = D_i g_i + f
\end{aligned} \tag{1.1}
$$

in $(-\infty, T) \times \mathbb{R}^d_+$, as well as the corresponding elliptic equations:

$$
\begin{aligned}
&a^{ij} D_{ij} u + b^i D_i u + cu - \lambda u = f, \\
&D_i\big(a^{ij} D_j u + b^i u\big) + \hat{b}^i D_i u + cu - \lambda u = D_i g_i + f
\end{aligned}
$$

in $\mathbb{R}^d_+$, where $\mathbb{R}^d_+ = \{x = (x_1, x') \in \mathbb{R}^d, x_1 > 0, x' \in \mathbb{R}^{d-1}\}$ and λ is a non-negative number. We consider the equations in the weighted Sobolev spaces $H^\gamma_{p,\theta}(\mathbb{R}^d_+)$ and $\mathbb{H}^\gamma_{p,\theta}((-\infty, T) \times \mathbb{R}^d_+)$, which were introduced in a unified manner by N.V. Krylov [21] for all $\gamma \in \mathbb{R}$. In particular, if γ is a non-negative integer,

$$
H^\gamma_{p,\theta} = H^\gamma_{p,\theta}\big(\mathbb{R}^d_+\big) = \big\{u : x_1^{|\alpha|} D^\alpha u \in L_{p,\theta}\big(\mathbb{R}^d_+\big)\ \forall \alpha : 0 \le |\alpha| \le \gamma\big\},
$$

where $L_{p,\theta}(\mathbb{R}^d_+)$ is an L_p space with the measure $\mu_d(dx) = x_1^{\theta-d}\, dx$.

Since the work in [21], much attention has been paid to the solvability theory for equations in the weighted Sobolev spaces $H^\gamma_{p,\theta}$; see [14,19,16,18]. The necessity of such theory came from stochastic partial differential equations (SPDEs) and is well explained in [20]. For SPDEs in weighted Sobolev spaces, we refer the reader to [28,13,12,26,17].

In this paper we extend the existing theory for equations in the weighted Sobolev spaces to a considerably more general setting. Compared to the known results in the literature, the features of our results can be summarized as follows:

- The leading coefficients a^{ij} are in a substantially larger class of functions.
- In the divergence case, the space of data (or *free terms*) is larger.
- The lower-order coefficients are not required to approach zero as $x_1 \to +\infty$.

The most significant difference from the previous results is that we allow the leading coefficients a^{ij} to be merely measurable in x_1-direction. That is, we do not assume any regularity conditions on a^{ij} as functions of x_1 variable. In the parabolic case, we further allow all the leading coefficients $a^{ij}(t, x)$ to be merely measurable in (t, x_1) except $a^{11}(t, x)$, which is either measurable in t or in x_1. As functions of the other variables, the coefficients a^{ij} have small bounded mean oscillations (BMO) (see assumptions in Section 2).

In the literature, the Laplace and heat equations in the weighted Sobolev spaces $H^\gamma_{p,\theta}$ were first considered in [21], when θ is in the optimal range $(d-1, d-1+p)$. These results