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Representations of rational Cherednik algebras with minimal support and torus knots

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ABSTRACT

In this paper we obtain several results about representations of rational Cherednik algebras, and discuss their applications. Our first result is the Cohen–Macaulayness property (as modules over the polynomial ring) of Cherednik algebra modules with minimal support. Our second result is an explicit formula for the character of an irreducible minimal support module in type A_{n-1} for $c = \frac{m}{n}$, and an expression of its quasispherical part (i.e., the isotypic part of "hooks") in terms of the HOMFLY polynomial of a torus knot colored by a Young diagram. We use this formula and the work of Calaque, Enriquez and Etingof to give explicit formulas for the characters of the irreducible equivariant D-modules on the nilpotent cone for SL_m. Our third result is the construction of the Koszul–BGG complex for the rational Cherednik algebra, which generalizes the construction of the Koszul–BGG resolution from [3] and [21], and the calculation of its homology in type A. We also show in type A that the differentials in the Koszul–BGG complex are uniquely determined by the condition that they are nonzero homomorphisms of modules over the Cherednik algebra.

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Finally, our fourth result is the symmetry theorem, which identifies the quasispherical components in the representations with minimal support over the rational Cherednik algebras $H_{\frac{m}{n}}(S_n)$ and $H_{\frac{n}{m}}(S_m)$. In fact, we show that the simple quotients of the corresponding quasispherical subalgebras are isomorphic as filtered algebras. This symmetry was essentially established in [8] in the spherical case, and in [24] in the case $GCD(m,n)=1$, and it has a natural interpretation in terms of invariants of torus knots.

1. Introduction

The goal of this paper is to establish a number of properties of representations of rational Cherednik algebras with minimal support, and connect them to knot invariants. Our motivation came from the connections of representations of Cherednik algebras with quantum invariants of torus knots and Hilbert schemes of plane curve singularities (such as $x^m = y^n$, $GCD(m,n)=1$), see [25,47].

1.1. Let $\mathfrak{h}$ be a finite dimensional complex vector space, $W \subset GL(\mathfrak{h})$ a finite subgroup, $S \subset W$ the set of reflections, and $c : S \to \mathbb{C}$ a conjugation invariant function. Let $H_c(W,\mathfrak{h})$ be the rational Cherednik algebra attached to $W, \mathfrak{h}$. Let $\mathcal{O}_c = \mathcal{O}_c(W,\mathfrak{h})$ be the category of modules over this algebra which are finitely generated over $\mathbb{C}[\mathfrak{h}] = S\mathfrak{h}^*$ and locally nilpotent under $\mathfrak{h}$. Typical examples of objects of this category are $M_c(\tau)$, the Verma (a.k.a. standard) module over $H_c(W,\mathfrak{h})$ with lowest weight $\tau \in \operatorname{Irrep} W$, and $L_c(\tau)$, the irreducible quotient of $M_c(\tau)$.

Any object $M \in \mathcal{O}_c$ is a finitely generated $\mathbb{C}[\mathfrak{h}]$-module. So we can define its support $\operatorname{supp}(M)$, which is a closed subvariety of $\mathfrak{h}$.

Definition 1.1. We say that $M \in \mathcal{O}_c$ has minimal support if no subset of $\operatorname{supp}(M)$ of smaller dimension is the support of a nonzero object of $\mathcal{O}_c$.

Our first main result is

Theorem 1.2. *If M has minimal support then it is a Cohen–Macaulay module over $\mathbb{C}[\mathfrak{h}]$ of dimension $d = \dim supp(M)$. In other words, it is free over any polynomial subalgebra $\mathbb{C}[p_1,\dots,p_d] \subset \mathbb{C}[\mathfrak{h}]$ (with homogeneous p_i) over which it is finitely generated.*

Remark 1.3. Note that the minimal support condition is needed. For example, if $W = S_3$, $c = 1/3$, and $M = L_c(\mathfrak{h})$ is the irreducible module with lowest weight $\mathfrak{h}$, then M is the augmentation ideal in $\mathbb{C}[\mathfrak{h}]$, so it is not Cohen–Macaulay (as it is not free).

1.2. Our second result is the character formula for irreducible minimally supported modules for rational Cherednik algebras of S_n for $c = \frac{m_0}{n_0}$, and its consequences. Let