Advances in Mathematics 277 (2015) 181–251

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

On convex projective manifolds and cusps

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A R T I C L E   I N F O

Article history:
Received 12 February 2015
Accepted 16 February 2015
Available online 27 March 2015
Communicated by Tomasz S. Mrowka

MSC:
57N16
57M50

Keywords:
Convex projective
Manifold
Cusp
Hyperbolic
Horosphere

A B S T R A C T

This study of properly or strictly convex real projective manifolds introduces notions of *parabolic, horosphere* and *cusp*. Results include a Margulis lemma and in the strictly convex case a thick–thin decomposition. Finite volume cusps are shown to be projectively equivalent to cusps of hyperbolic manifolds. This is proved using a characterization of ellipsoids in projective space.
Except in dimension 3, there are only finitely many topological types of strictly convex manifolds with bounded volume. In dimension 4 and higher, the diameter of a closed strictly convex manifold is at most 9 times the diameter of the thick part. There is an algebraic characterization of strict convexity in terms of relative hyperbolicity.

Surfaces are ubiquitous throughout mathematics; in good measure because of the *geometry* of Riemann surfaces. Similarly, Thurston's insights into the geometry of

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1 Cooper was supported in part by NSF grants DMS-0706887, 1065939, 1045292.
2 Long was supported in part by NSF grant DMS-1005659.
3 Tillmann was supported in part by ARC grant DP1095760.

http://dx.doi.org/10.1016/j.aim.2015.02.009

3-manifolds have led to many developments in diverse areas. This paper develops the bridge between real projective geometry and low dimensional topology.

Real projective geometry is a rich subject with many connections. In recent years it has been combined with topology in the study of projective structures on manifolds. Classically it provides a unifying framework as it contains the three constant curvature geometries as subgeometries. In dimension 3 it contains the eight Thurston geometries (up to a subgroup of index 2 in the case of product geometries) and there are paths of projective structures that correspond to transitions between different Thurston geometries on a fixed manifold. Moreover, there is a link between real projective deformations and complex hyperbolic deformations of a real hyperbolic orbifold (see [23]). Projective geometry therefore offers a general and versatile viewpoint for the study of 3-manifolds.

Another window to projective geometry: The symmetric space $SL(n,\mathbb{R})/SO(n)$ is isomorphic to the group of projective automorphisms of the convex set in projective space obtained from the open cone of positive definite quadratic forms in n variables. This set is *properly convex*: its closure is a compact convex set, which is disjoint from some projective hyperplane. The boundary of the closure has a rich structure as it consists of semi-definite forms and, when $n = 3$, contains a dense set of flat 2-discs; each corresponding to a family of semi-definite forms of rank 2 which may be identified with a copy of the hyperbolic plane.

From a geometrical point of view there is a crucial distinction between *strictly convex* domains, which contain no straight line segment in the boundary, and the more general class of properly convex domains. The former behave like manifolds of negative sectional curvature and the latter like arbitrary symmetric spaces. However, projective manifolds are more general: Kapovich [37] has shown that there are closed strictly convex 4-manifolds which do not admit a hyperbolic structure.

The *Hilbert metric* is a complete Finsler metric on a properly convex set Ω. This is the hyperbolic metric in the Klein model when Ω is a round ball. A simplex with the Hilbert metric is isometric to a normed vector space, and appears in a natural geometry on the Lie algebra $\mathfrak{sl}_n$. A singular version of this metric arises in the study of certain limits of projective structures. The Hilbert metric has a Hausdorff *measure* and hence a notion of *finite volume*.

If a manifold of dimension greater than 2 admits a finite volume complete hyperbolic metric, then by Mostow–Prasad rigidity that metric is unique up to isometry. In dimension 2 there is a finite dimensional Teichmüller space of deformations, parameterized by an algebraic variety. In the context of strictly convex structures on *closed* manifolds the deformation space is a semi-algebraic variety. There are closed hyperbolic 3-manifolds for which this deformation space has arbitrarily large dimension. Part of the motivation for this work is to extend these ideas to the context of finite volume structures, which in turn is motivated by the study of these (and other still mysterious) examples which arise via deformations of some finite volume non-compact convex projective 3-orbifolds. (See [22] and [23].)