Advances in Mathematics 277 (2015) 252–282

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

A new approach to constant term identities and Selberg-type integrals

Gyula Károlyi [a,*], Zoltán Lóránt Nagy [a], Fedor V. Petrov [b,1,2], Vladislav Volkov [c,2]

[a] MTA Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, Budapest, 1053, Hungary
[b] Steklov Institute of Mathematics, Fontanka 27, 191023 St. Petersburg, Russia
[c] Saint-Petersburg State University, Universitetsky prospekt 28, 198504 St. Petersburg, Russia

ARTICLE INFO

Article history:
Received 5 March 2014
Accepted 30 September 2014
Available online 27 March 2015
Communicated by Gil Kalai

Keywords:
Aomoto's constant term identity
Calogero–Sutherland model
Combinatorial Nullstellensatz
Erdős–Heilbronn conjecture
Forrester's conjecture
Hermite interpolation
Selberg integral

ABSTRACT

Selberg-type integrals that can be turned into constant term identities for Laurent polynomials arise naturally in conjunction with random matrix models in statistical mechanics. Built on a recent idea of Karasev and Petrov we develop a general interpolation based method that is powerful enough to establish many such identities in a simple manner. The main consequence is the proof of a conjecture of Forrester related to the Calogero–Sutherland model. In fact we prove a more general theorem, which includes Aomoto's constant term identity at the same time. We also demonstrate the relevance of the method in additive combinatorics.

* Corresponding author. Also at the Institute of Mathematics, Eötvös University, Budapest. Visiting the School of Mathematics and Physics at the University of Queensland.

E-mail addresses: karolyi@cs.elte.hu (G. Károlyi), nagyzoltanlorant@gmail.com (Z.L. Nagy), fedyapetrov@gmail.com (F.V. Petrov), vladvolkov239@gmail.com (V. Volkov).

[1] Also at St. Petersburg State University.

[2] Also at B.N. Delone International Laboratory "Discrete and Computational Geometry".

http://dx.doi.org/10.1016/j.aim.2014.09.028

1. Introduction

Perhaps the most famous constant term identity is the one associated with the name of Freeman Dyson. In his seminal paper [13] dated back to 1962, Dyson proposed to replace Wigner's classical Gaussian-based random matrix models by what now is known as the circular ensembles. The study of their joint eigenvalue probability density functions led Dyson to the following conjecture. Consider the family of Laurent polynomials

$$\mathcal{D}(\boldsymbol{x};\boldsymbol{a}) := \prod_{1\le i\ne j\le n}\left(1-\frac{x_i}{x_j}\right)^{a_i}$$

parametrized by a sequence $\boldsymbol{a} = (a_1,\dots,a_n)$ of nonnegative integers, where $\boldsymbol{x} = (x_1,\dots,x_n)$ is a sequence of indeterminates. Denoting by $\mathrm{CT}[\mathcal{L}(\boldsymbol{x})]$ the constant term of the Laurent polynomial $\mathcal{L} = \mathcal{L}(\boldsymbol{x})$, Dyson's hypothesis can be formulated as the identity

$$\mathrm{CT}[\mathcal{D}(\boldsymbol{x};\boldsymbol{a})] = \frac{(a_1+a_2+\cdots+a_n)!}{a_1!a_2!\dots a_n!} =: \binom{|\boldsymbol{a}|}{\boldsymbol{a}},$$

where $|\boldsymbol{a}| = a_1+a_2+\cdots+a_n$. Using the shorthand notation $\mathcal{D}(\boldsymbol{x};k)$ for the equal parameter case $\boldsymbol{a} = (k,\dots,k)$, the constant term of $\mathcal{D}(\boldsymbol{x};k)$ for $k = 1,2,4$ corresponds to the normalization factor of the partition function for the circular orthogonal, unitary and symplectic ensemble, respectively.

Dyson's conjecture was confirmed by Gunson [unpublished][3] and Wilson [49] in the same year. The most elegant proof, based on Lagrange interpolation, is due to Good [22].

Let q denote yet another independent variable. In 1975 Andrews [4] suggested the following q-analogue of Dyson's conjecture: The constant term of the Laurent polynomial

$$\mathcal{D}_q(\boldsymbol{x};\boldsymbol{a}) := \prod_{1\le i<j\le n}\left(\frac{x_i}{x_j}\right)_{a_i}\left(\frac{qx_j}{x_i}\right)_{a_j}$$

must be the q-multinomial coefficient

$$\begin{bmatrix}|\boldsymbol{a}|\\ \boldsymbol{a}\end{bmatrix} := \frac{(q)_{|\boldsymbol{a}|}}{(q)_{a_1}(q)_{a_2}\dots(q)_{a_n}},$$

where $(t)_k = (1-t)(1-tq)\dots(1-tq^{k-1})$. Note that the slight asymmetry of the function $\mathcal{D}_q$ disappears when one considers $\mathcal{D} = \mathcal{D}_1$; specializing at $q = 1$, Andrews' conjecture gives back that of Dyson.

[3] Gunson's proof is similar to that of Wilson, cf. [13]. A related conjecture of Dyson is proved by Gunson in [23].