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Arc spaces and the vertex algebra commutant problem

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A B S T R A C T

Given a vertex algebra $\mathcal{V}$ and a subalgebra $\mathcal{A} \subset \mathcal{V}$, the commutant $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ is the subalgebra of $\mathcal{V}$ which commutes with all elements of $\mathcal{A}$. This construction is analogous to the ordinary commutant in the theory of associative algebras, and is important in physics in the construction of coset conformal field theories. When $\mathcal{A}$ is an affine vertex algebra, $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ is closely related to rings of invariant functions on arc spaces. We find strong finite generating sets for a family of examples where $\mathcal{A}$ is affine and $\mathcal{V}$ is a $\beta\gamma$-system, bc-system, or $bc\beta\gamma$-system.

1. Introduction

Let $\mathcal{V}$ be a vertex algebra, and let $\mathcal{A}$ be a subalgebra of $\mathcal{V}$. The *commutant* of $\mathcal{A}$ in $\mathcal{V}$, denoted by $\mathrm{Com}(\mathcal{A}, \mathcal{V})$, is the subalgebra consisting of all elements $v \in \mathcal{V}$ such that

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$[a(z), v(w)] = 0$ for all $a \in \mathcal{A}$. This construction was introduced by Frenkel and Zhu in [6], generalizing earlier constructions in representation theory [11] and physics [8], and is important in the construction of coset conformal field theories. It is also natural to study the double commutant $\mathrm{Com}(\mathrm{Com}(\mathcal{A}, \mathcal{V}), \mathcal{V})$, which always contains $\mathcal{A}$. If $\mathcal{A} = \mathrm{Com}(\mathrm{Com}(\mathcal{A}, \mathcal{V}), \mathcal{V})$, we say that $\mathcal{A}$ and $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ form a *Howe pair* inside $\mathcal{V}$. If $\mathcal{A}$ acts semisimply on $\mathcal{V}$, $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ can be studied by decomposing $\mathcal{V}$ as an $\mathcal{A}$-module. Otherwise, there are few existing techniques for studying commutants, and there are very few examples where an exhaustive description can be given in terms of generators, operator product expansions, and normally ordered polynomial relations among the generators.

An equivalent definition of $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ is the set of elements $v \in \mathcal{V}$ such that $a \circ_n v = 0$ for all $a \in \mathcal{A}$ and $n \geq 0$. We may regard $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ as the algebra of invariants in $\mathcal{V}$ under the action of $\mathcal{A}$. If $\mathcal{A}$ is a homomorphic image of an affine vertex algebra associated to some Lie algebra $\mathfrak{g}$, $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ is just the invariant space $\mathcal{V}^{\mathfrak{g}[t]}$, and in this case one can apply techniques from invariant theory and commutative algebra. This approach was introduced in [15], and the structure that makes it work is a *good increasing filtration* on $\mathcal{V}$. The associated graded object $\mathrm{gr}(\mathcal{V})$ is then an abelian vertex algebra, i.e., a (super)commutative ring with a differential, and in many cases it can be interpreted as the ring $\mathcal{O}(X_\infty)$ of functions on the arc space X_∞ of some scheme X. For example, the level k universal affine vertex algebra $V_k(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ has a filtration for which $\mathrm{gr}(V_k(\mathfrak{g})) \cong \mathcal{O}(\mathfrak{g}_\infty)$ and the $\beta\gamma$-system $\mathcal{S}(V)$ of a finite-dimensional vector space V has $\mathrm{gr}(\mathcal{S}(V)) \cong \mathcal{O}((V \oplus V^*)_\infty)$. For any X, $\mathcal{O}(X_\infty)$ is an abelian vertex algebra [4], and one of the themes of this paper is that the geometry of arc spaces can be used to answer structural questions about nonabelian vertex algebras as well.

If $\mathcal{A} \subset \mathcal{V}$ is an affine vertex algebra and $\mathcal{V}$ has a $\mathfrak{g}[t]$-invariant good increasing filtration, there is an action of $\mathfrak{g}[t]$ on $\mathrm{gr}(\mathcal{V})$ by derivations of degree zero. There is an injective map of differential (super)commutative algebras

$$\mathrm{gr}(\mathcal{V}^{\mathfrak{g}[t]}) \hookrightarrow \mathrm{gr}(\mathcal{V})^{\mathfrak{g}[t]}. \tag{1.1}$$

Unfortunately, the associated graded functor and the invariant functor need not commute with each other, and this map is generally not an isomorphism. The structure of $\mathrm{gr}(\mathcal{V})^{\mathfrak{g}[t]}$ is simpler than that of $\mathrm{gr}(\mathcal{V}^{\mathfrak{g}[t]})$, and is closely related to rings of invariant functions on arc spaces. If $\mathrm{gr}(\mathcal{V})^{\mathfrak{g}[t]}$ is finitely generated as a differential algebra (which is the case in our main examples), checking the surjectivity of (1.1) becomes a finite problem. If (1.1) is surjective, there is a reconstruction theorem that says that a generating set for $\mathrm{gr}(\mathcal{V})^{\mathfrak{g}[t]}$ as a differential algebra corresponds to a strong generating set for $\mathcal{V}^{\mathfrak{g}[t]}$ as a vertex algebra. Moreover, all normally ordered polynomial relations among the generators of $\mathcal{V}^{\mathfrak{g}[t]}$ correspond to classical relations in $\mathrm{gr}(\mathcal{V})^{\mathfrak{g}[t]}$, with suitable quantum corrections.

In our main examples, $\mathcal{V}$ is the $\beta\gamma$-system $\mathcal{S}(V)$, which is the vertex algebra analogue of the Weyl algebra $\mathcal{D}(V)$. If G is a connected, reductive Lie group with Lie algebra $\mathfrak{g}$