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Deformation of Gabor systems ☆

Karlheinz Gröchenig [a,*], Joaquim Ortega-Cerdà [b], José Luis Romero [a]

[a] *Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria*
[b] *Dept. Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain*

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We introduce a new notion for the deformation of Gabor systems. Such deformations are in general nonlinear and, in particular, include the standard jitter error and linear deformations of phase space. With this new notion we prove a strong deformation result for Gabor frames and Gabor Riesz sequences that covers the known perturbation and deformation results. Our proof of the deformation theorem requires a new characterization of Gabor frames and Gabor Riesz sequences. It is in the style of Beurling's characterization of sets of sampling for bandlimited functions and extends significantly the known characterization of Gabor frames "without inequalities" from lattices to non-uniform sets.

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* Corresponding author.
E-mail addresses: karlheinz.groechenig@univie.ac.at (K. Gröchenig), jortega@ub.edu (J. Ortega-Cerdà), jose.luis.romero@univie.ac.at (J.L. Romero).

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Stability estimate
Weak limit

1. Introduction

The question of robustness of a basis or frame is a fundamental problem in functional analysis and in many concrete applications. It has its historical origin in the work of Paley and Wiener (see, e.g., [40]) who studied the perturbation of Fourier bases and was subsequently investigated in many contexts in complex analysis and harmonic analysis. Particularly fruitful was the study of the robustness of structured function systems, such as reproducing kernels, sets of sampling in a space of analytic functions, wavelets, or Gabor systems. In this paper we take a new look at the stability of Gabor frames and Gabor Riesz sequences with respect to general deformations of phase space.

To be explicit, let us denote the time–frequency shift of a function $g \in L^2(\mathbb{R}^d)$ along $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d}$ by

$$\pi(z)g(t) = e^{2\pi i \xi \cdot t} g(t - x) .$$

For a fixed non-zero function $g \in L^2(\mathbb{R}^d)$, usually called a “window function”, and $\Lambda \subseteq \mathbb{R}^{2d}$, a Gabor system is a structured function system of the form

$$\mathcal{G}(g, \Lambda) = \left\{ \pi(\lambda)g := e^{2\pi i \xi \cdot} g(\cdot - x) : \lambda = (x, \xi) \in \Lambda \right\} .$$

The index set Λ is a discrete subset of the phase space $\mathbb{R}^{2d}$ and λ indicates the localization of a time–frequency shift $\pi(\lambda)g$ in phase space.

The Gabor system $\mathcal{G}(g, \Lambda)$ is called a *frame* (a Gabor frame), if

$$A \|f\|_2^2 \le \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \le B \|f\|_2^2 , \qquad f \in L^2(\mathbb{R}^d),$$

for some constants $0 < A \le B < \infty$. In this case every function $f \in L^2(\mathbb{R}^d)$ possesses an expansion $f = \sum_\lambda c_\lambda \pi(\lambda) g$, for some coefficient sequence $c \in \ell^2(\Lambda)$ such that $\|f\|_2 \asymp \|c\|_2$. The Gabor system $\mathcal{G}(g, \Lambda)$ is called a *Riesz sequence* (or Riesz basis for its span), if $\|\sum_\lambda c_\lambda \pi(\lambda) g\|_2 \asymp \|c\|_2$ for all $c \in \ell^2(\Lambda)$.

For meaningful statements about Gabor frames it is usually assumed that

$$\int_{\mathbb{R}^{2d}} |\langle g, \pi(z)g \rangle| \, dz < \infty.$$

This condition describes the modulation space $M^1(\mathbb{R}^d)$, also known as the Feichtinger algebra. Every Schwartz function satisfies this condition.

In this paper we study the stability of the spanning properties of $\mathcal{G}(g, \Lambda)$ with respect to a set $\Lambda \subseteq \mathbb{R}^{2d}$. If Λ' is “close enough” to Λ, then we expect $\mathcal{G}(g, \Lambda')$ to possess