Advances in Mathematics 271 (2015) 1–29

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Thin sums matroids and duality

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ARTICLE INFO

Article history:
Received 7 May 2013
Accepted 21 October 2014
Available online 3 December 2014
Communicated by Andreas Dress

Keywords:
Infinite matroid
Thin sums
Matroid with coefficients

ABSTRACT

Building on the recent axiomatisation of infinite matroids with duality, we present a theory of representability for infinite matroids. This notion of representability allows for infinite sums, and is preserved under duality.

1. Introduction

Central to the basic theory of finite matroids is the fact that they can be axiomatised in many different ways. This includes axiomatisations in terms of independent sets, bases, circuits, closure operators and rank functions. However, when applied to infinite sets, the different axiomatisations are no longer mutually interpretable, leading to the question of whether there is a natural theory of infinite matroids. One solution is to restrict attention to *finitary* matroids: those in which a set is independent just when all of its finite subsets are, or equivalently in which all circuits are finite. By adding suitable finitariness constraints, most of the axiomatisations can be once more rendered equivalent. Much of the basic theory of finite matroids can be extended to this class via compactness arguments. A serious problem with this approach is that it does not provide

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http://dx.doi.org/10.1016/j.aim.2014.10.010

a good theory of duality, which is one of the central elements of finite matroid theory. A finitary matroid needs not have a finitary dual.

This led Rado to ask in 1966 whether there is a good theory of infinite matroids with duality [13, Problem P531]. A wide range of matroid-like structures with duality were proposed, but no canonical answer emerged and for decades it was believed that Rado's question might not have a sensible answer. Recently Bruhn et al. [8], continuing work of Higgs [11] and Oxley [12], presented a way to extend five of the axiomatisations of finite matroids to mutually interpretable axiomatisations of one of these proposed classes of structure. The simplicity of these axiomatisations, together with the rapid theoretical progress to which they have led (some of which is summarised below, but see [8] for a more detailed account), strongly suggests that they provide the correct approach to the theory of infinite matroids.

Even with a sensible theory of infinite matroids in place, many basic conceptual questions remain. The question addressed by this paper is that of how to extend the notion of representability over a field from finitary to non-finitary matroids. If we have a (possibly infinite) family of vectors in a vector space over some field k, we get a matroid structure on that family whose independent sets are given by the linearly independent subsets of the family. Matroids arising in this way are called *representable* matroids over k, and are always finitary.

Although many interesting finite matroids (e.g. all graphic matroids) are representable, many interesting examples of infinite matroids cannot be of this type, because they are not finitary. Another problem is that in restricting attention to finitary matroids we would once more lose the power of duality: if a finite matroid is representable over the field k then so is its dual, but the dual of an infinite matroid representable over k need not be finitary. So there is a question here akin to Rado's question:

Question 1.1. *Is there a good theory of infinite matroids representable over k with duality?*

Bruhn and Diestel explored one approach to this question in [7]. They tried extending the notion of linear combinations to allow for infinite combinations in certain constrained circumstances.

The construction relies on taking the vector space to be of the form k^A for some set A. We allow linear combinations of infinitely many vectors. However, we require these linear combinations to be well defined pointwise. This means that for each $a \in A$ there are only finitely many nonzero coefficients at vectors with nonzero component at a. Further details are given in Section 2. Sadly, it turns out that there are examples of systems of independent sets definable in this way which are not matroids. Accordingly, we refer to such systems in general as *thin sums systems*, and only call them thin sums matroids if they really are matroids. Thin sums matroids need not be finitary.

Because thin sums systems are often not matroids, Bruhn and Diestel focused on a class of thin sums systems which they were able to show are matroids [7], namely those generated from families of vectors in which for each $a \in A$ there are only finitely many