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High density piecewise syndeticity of sumsets

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Renling Jin proved that if A and B are two subsets of the natural numbers with positive Banach density, then $A+B$ is piecewise syndetic. In this paper, we prove that, under various assumptions on positive lower or upper densities of A and B, there is a high density set of witnesses to the piecewise syndeticity of $A+B$. Most of the results are shown to hold more generally for subsets of $\mathbb{Z}^d$. The key technical tool is a Lebesgue density theorem for measure spaces induced by cuts in the nonstandard integers.

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Syndeticity
Nonstandard analysis

1. Introduction and preliminaries

1.1. Sumsets and piecewise syndeticity

The earliest result on the relationships between density of sequences, sum or difference sets, and syndeticity is probably Furstenberg's theorem mentioned in [7, Proposition 3.19]: If A has positive upper Banach density, then $A-A$ is syndetic, i.e. has bounded gaps. The proof of the theorem is essentially a pigeonhole argument.

In [9] Jin shows that if A and B are two subsets of $\mathbb{N}$ with positive upper Banach densities, then $A+B$ must be piecewise syndetic, i.e. for some m, $A+B+[0,m]$ contains arbitrarily long intervals. Jin's proof uses nonstandard analysis. In [11], this result is extended to abelian groups with tiling structures. In [9,11] the question as to whether this result can be extended to any countable amenable group is posed, and in [2] a positive answer to the above question is proven. It is shown that if A and B are two subsets of a countable amenable group with positive upper Banach densities, then $A \cdot B$ is piecewise Bohr, which implies piecewise syndeticity. In fact, a stronger theorem is obtained in the setting of countable abelian groups: A set S is piecewise Bohr if and only if S contains the sum of two sets A and B with positive upper Banach densities. Jin's theorem was generalized to arbitrary amenable groups in [5]. At the same time, several new proofs of the theorem in [9] have appeared. For example, an ultrafilter proof is obtained in [1]. A more quantitative proof that includes a bound based on the densities is obtained in [4] by nonstandard methods, and in [3] by elementary means.

However, there has not been any progress on extending the theorem in [9] to lower asymptotic density or upper asymptotic density instead of upper Banach density. Of course, if A and B have positive lower (upper) asymptotic densities then they have positive Banach density, so $A+B$ must be piecewise syndetic. In this paper we show that there is significant uniformity to the piecewise syndeticity in the sense that there are a large density of points in the sumset with no gap longer than some fixed m. Furthermore, this can be extended to all finite dimensions. Specifically we show the following:

Theorem 1. *Suppose that A and B are subsets of $\mathbb{Z}^d$. For $m,k \in \mathbb{N}$, set*

$$S_{m,k}(A,B) := \{z \in \mathbb{Z}^d \ : \ z+[-k,k]^d \subseteq A+B+[-m,m]^d\}.$$

Then:

1. *If A has positive upper density α and B has positive Banach density, then there exists an m such that, for all k, $S_{m,k}(A,B)$ has upper density at least α (Theorem 14).*