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Hall–Littlewood polynomials and vector bundles on the Hilbert scheme

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ABSTRACT

Let E be the bundle defined by applying a polynomial functor to the tautological bundle on the Hilbert scheme of n points in the complex plane. By a result of Haiman [5], the Čech cohomology groups $H^i(E)$ vanish for all $i > 0$. It follows that the equivariant Euler characteristic with respect to the standard two-dimensional torus action has nonnegative integer coefficients in the torus variables z_1, z_2, because they count the dimensions of the weight spaces of $H^0(E)$. We derive a formula for this Euler characteristic using residue formulas for the Euler characteristic coming from the description of the Hilbert scheme as a quiver variety [13,14]. We evaluate this expression using Jing's Hall–Littlewood vertex operator with parameter z_1 [7], and a new vertex operator formula given in Proposition 1 below. We conjecture that the summand in this formula is a polynomial in z_1 with nonnegative integer coefficients, a special case of which was known to Lascoux and Schützenberger [8].

1. Introduction

Let $\mathrm{Hilb}_n\, \mathbb{C}^2$ denote the Hilbert scheme of n points in the complex plane, and consider the standard two-dimensional torus action on it induced from the planar action

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$$T = (\mathbb{C}^*)^2 \circlearrowright \mathbb{C}^2, \quad (z_1, z_2) \cdot (x, y) = (z_1^{-1}x, z_2^{-1}y) \tag{1}$$

by pullback of ideals. We also have an n-dimensional tautological bundle $\mathcal{U}$ on the Hilbert scheme, whose fiber over a subscheme Z defining a point in $\mathrm{Hilb}_n \mathbb{C}^2$ is simply the space of sections of $\mathcal{O}_Z$, and which inherits an action of T, see [10] for details. Given a polynomial representation ρ of GL_n, we obtain a new equivariant bundle $E = \rho(\mathcal{U})$, whose fibers are given by applying ρ to the fibers of $\mathcal{U}$. For instance, if ρ is given by the kth exterior power of the defining representation of GL_n, then $E = \Lambda^k\mathcal{U}$ is the kth exterior power of $\mathcal{U}$. If ρ_μ is the irreducible polynomial representation of GL_n corresponding to a partition μ, then by Schur–Weyl duality $\rho_\mu(\mathcal{U})$ is the subbundle of $\mathcal{U}^{\boxtimes k}$ associated to the irreducible subrepresentation χ_μ of S_k for $k = |\mu|$, which acts by permuting the factors. The corresponding functor is called the Schur functor.

We may consider the Čech cohomology groups $H^i(E)$, as well as the equivariant Euler characteristic

$$\chi_n(E) = \sum_i (-1)^i \operatorname{ch} H^i_{\mathrm{Hilb}_n}(E) \in \mathbb{Z}[[z_1^{\pm 1}, z_2^{\pm 1}]],$$

where ch denotes the character of $H^i(E)$ as a representation of T. If Λ is the ring of symmetric polynomials in infinitely many variables, the polynomial representations are in the image of the map

$$\Lambda \to K_T(\mathrm{Hilb}_n \mathbb{C}^2), \quad s_\mu \mapsto \mathbb{S}_\mu(\mathcal{U}),$$

where $s_\mu \in \Lambda$ is the Schur polynomial, $\mathbb{S}_\mu$ is the corresponding representation of GL_n (the Schur functor), and μ is a partition. Since the Euler characteristic is defined at the level of K-theory, we have a well defined Euler characteristic $\chi_n(f(\mathcal{U}))$, for any symmetric function $f \in \Lambda$.

The main result of this paper is the following:

Theorem A. *The Euler characteristic is given by*

$$\chi_n(f(\mathcal{U})) = \sum_{\mu,\nu} z_2^{|\mu|} z_1^{|\mu|+k_{\mu\nu}} b_{\nu,n}(z_1)^{-1} f_{\nu\mu}(z_1).$$

Here $k_{\mu\nu}$ is an integer, $b_{\nu,n}(z)$ is the norm squared of the Hall–Littlewood polynomial $P_\nu(X; z)$ in n variables, and $f_{\nu\mu}(z)$ is the matrix element of the operator of multiplication by f in the Hall–Littlewood basis.

We conjecture that the coefficients of the polynomial $f_{\nu\mu}(z)$ are *nonnegative* integers whenever f defines an honest representation, i.e. is a nonnegative integral linear combination of Schur polynomials. This positivity is a combinatorial manifestation of a result of Haiman which says that the Čech cohomology groups $H^i(\mathcal{U}^{\otimes l} \otimes P)$ vanish for $i > 0$, where P is the Procesi bundle [5]. Since the trivial bundle is a summand of the Procesi