Advances in Mathematics 278 (2015) 67–102

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Lagrangian–Eulerian methods for uniqueness in hydrodynamic systems

Peter Constantin

Department of Mathematics, Princeton University, Princeton, NJ 08544, United States

ARTICLE INFO

Article history:
Received 31 July 2014
Accepted 26 March 2015
Available online 11 April 2015
Communicated by Charles Fefferman

Keywords:
Stokes equations
Navier–Stokes equations
Lagrangian methods
Uniqueness
Oldroyd-B
Magnetohydrodynamics

ABSTRACT

We present a Lagrangian–Eulerian strategy for proving uniqueness and local existence of solutions of limited smoothness for a class of incompressible hydrodynamic models including Oldroyd-B type complex fluid models and zero magnetic resistivity magneto-hydrodynamics equations.

1. Introduction

Many physical models consist of equations for fluids coupled with equations for other fields. Primary examples occur in descriptions of complex fluids in which a solvent interacts with particles, and in magneto-hydrodynamics, in which a fluid interacts with a magnetic field. One of the simplest complex fluids models, an Oldroyd-B model, reduces to a time independent Stokes system

$$-\Delta u + \nabla p = \operatorname{div} \sigma, \quad \operatorname{div} u = 0$$

E-mail address: const@math.princeton.edu.

http://dx.doi.org/10.1016/j.aim.2015.03.010

coupled with an evolution equation for the symmetric added stress matrix σ,

$$\partial_t \sigma + u \cdot \nabla \sigma = (\nabla u)\sigma + \sigma(\nabla u)^T - \sigma + (\nabla u) + (\nabla u)^T$$

Clearly, from the Stokes equation with appropriate boundary conditions (for instance decay in the whole space) it follows that the velocity gradient is of the same order of magnitude as the added stress, $\nabla u \sim \sigma$. This makes the evolution equation for σ potentially capable of producing finite time blow up. The formation of finite time singularities in this system is an outstanding open problem. While the balance $\sigma \sim \nabla u$ is potentially dangerous for large data and long time, it also indicates clearly that if ∇u is controlled then σ is controlled as well. In particular, the short time existence of solutions can be obtained in a class of velocities that is close to the Lipschitz class. The fact that singular integral operators are not bounded in L^∞ requires the use of slightly smaller spaces, and $\sigma \in C^\alpha(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and correspondingly $u \in C^{1+\alpha}(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$ are spaces in which the problem admits short time existence. Here both $\alpha \in (0,1)$ and $p \in (1,\infty)$ are arbitrary. It is natural then to ask about uniqueness of solutions in the same spaces. Taking the difference σ between two solutions σ_1 and σ_2 leads to an equation

$$\partial_t \sigma + \bar{u} \cdot \nabla \sigma + u \cdot \nabla \bar{\sigma} = (\nabla \bar{u})\sigma + (\nabla u)\bar{\sigma} + \bar{\sigma}(\nabla u)^T + \sigma(\nabla \bar{u})^T - \sigma + (\nabla u) + (\nabla u)^T$$

where $u = u_1 - u_2$ is the difference of the corresponding velocities, and $\bar{u} = \frac{1}{2}(u_1 + u_2)$ and $\bar{\sigma} = \frac{1}{2}(\sigma_1 + \sigma_2)$ are the arithmetic averages of velocities and of stresses. The right hand side is well-behaved in C^α. The term $u \cdot \nabla \bar{\sigma}$ is not defined for $\bar{\sigma} \in C^\alpha$. The left hand side is a divergence, and this would suggest the use of negative index Sobolev spaces, but the right hand does not permit it. These obstructions make an Eulerian approach to a uniqueness proof difficult in this class of solutions. Uniqueness with this low regularity was proved in [1], using however a Lagrangian approach. The main reason why Lagrangian variables are better behaved than Eulerian ones is that in Lagrangian variables the velocity $v = u \circ X$ is obtained from the Lagrangian added stress $\tau = \sigma \circ X$ by an expression

$$v = \mathbb{U}(\tau \circ X^{-1}) \circ X$$

where $\mathbb{U}$ is the linear operator that produces the solution of the steady Stokes equation from the added stresses, and X is the Lagrangian path, which is a time-dependent diffeomorphism. The Gateaux derivative (variational derivative or first variation, in the language of mechanics) of the map $X \mapsto v$ is a commutator, and it is better behaved than each of its terms. On the other hand, τ obeys an ODE in Lagrangian variables, so it is easily controlled for short time by $g = (\nabla u) \circ X$.

The present paper expands this approach to time-dependent relationships between u and σ and we prove uniqueness and local existence in large spaces $C^\alpha \cap L^p$ for a class of hydrodynamic models including complex fluids of Oldroyd-B type, and ideal magneto-hydrodynamics. Local existence of very smooth solutions of such systems is