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Wave packet analysis of Schrödinger equations in analytic function spaces

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A B S T R A C T

We consider a class of linear Schrödinger equations in $\mathbb{R}^d$, with analytic symbols. We prove a global-in-time integral representation for the corresponding propagator as a generalized Gabor multiplier with a window analytic and decaying exponentially at infinity, which is transported by the Hamiltonian flow. We then provide three applications of the above result: the exponential sparsity in phase space of the corresponding propagator with respect to Gabor wave packets, a wave packet characterization of Fourier integral operators with analytic phases and symbols, and the propagation of analytic singularities.

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1. Introduction

Consider the Cauchy problem

$$\begin{cases} D_t u + a^w(t,x,D)u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $D_t = -i\partial_t$ and the real-valued symbol $a(t,x,\xi)$ is continuous in t in some interval $[0,T]$ and smooth with respect to $x, \xi \in \mathbb{R}^d$, satisfying

$$|\partial_z^\alpha a(t,z)| \leq C_\alpha, \quad |\alpha| \geq 2, \; z \in \mathbb{R}^{2d}, \; t \in [0,T] \tag{2}$$

(Weyl quantization is understood). As a typical model one can consider the case when $a(t,x,\xi)$ is a quadratic form in x, ξ, which gives rise to metaplectic operators. Equations of this type turned out to be important in spectral theory [26,27] and in regularization issues for equations with rough coefficients [47]. Depending on the applications, several additional conditions are imposed on the symbol a and its derivatives. In any case a fundamental problem is to obtain some integral representation of the propagator, from which one can then deduce estimates for the solutions. In principle, *for small time* one expects the propagator to be represented by a Fourier integral operator (FIO)

$$S(t,0)f = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\Phi(t,x,\eta)} \sigma(t,x,\eta) \, \widehat{f}(\eta) \, d\eta \tag{3}$$

with a smooth real-valued phase $\Phi(t,x,\xi)$, having quadratic growth with respect to the variables x, ξ, and a symbol $\sigma(t,\cdot) \in S^0_{0,0}$, i.e. bounded together with its derivatives. This was first proved in [7,26,27] for a class of symbols $a(t,x,\xi)$ of polyhomogeneous type, i.e. with an asymptotic expansion in homogeneous terms in $z = (x,\xi)$, of decreasing order. These decay conditions are essential for the symbolic calculus to work and the integral representation was in fact constructed by the WKB method. Recently in [10] the above representation (3) was proved to be true for small time only under the assumption (2). Such a representation however does no longer keep valid for large time because of the appearance of caustics; in other terms, the space of FIOs is not an algebra. Different approaches have been proposed by several authors, see e.g. [1,3–5,11,19,21,30,32,35,41,48,49].

In the case of analytic symbols, which is the framework of this paper, there are further technical difficulties and surprisingly, to our knowledge, it is not even known whether the *exact* integral representation (3) holds for an analytic phase and symbol, at least for small time. We will answer positively this question as a byproduct of more general results.

Namely, consider the analytic symbol classes $S_a^{(k)}$, $k \in \mathbb{N}$, defined by the estimates

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C^{|\alpha|+|\beta|+1} \alpha! \beta!, \quad |\alpha| + |\beta| \geq k, \; x, \xi \in \mathbb{R}^d, \tag{4}$$

endowed with the obvious inductive limit topology of Fréchet spaces.