

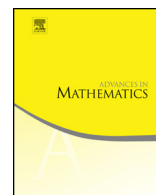


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# Wave packet analysis of Schrödinger equations in analytic function spaces



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## ABSTRACT

We consider a class of linear Schrödinger equations in  $\mathbb{R}^d$ , with analytic symbols. We prove a global-in-time integral representation for the corresponding propagator as a generalized Gabor multiplier with a window analytic and decaying exponentially at infinity, which is transported by the Hamiltonian flow. We then provide three applications of the above result: the exponential sparsity in phase space of the corresponding propagator with respect to Gabor wave packets, a wave packet characterization of Fourier integral operators with analytic phases and symbols, and the propagation of analytic singularities.

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## 1. Introduction

Consider the Cauchy problem

$$\begin{cases} D_t u + a^w(t, x, D)u = 0, \\ u(0) = u_0, \end{cases} \quad (1)$$

where  $D_t = -i\partial_t$  and the real-valued symbol  $a(t, x, \xi)$  is continuous in  $t$  in some interval  $[0, T]$  and smooth with respect to  $x, \xi \in \mathbb{R}^d$ , satisfying

$$|\partial_z^\alpha a(t, z)| \leq C_\alpha, \quad |\alpha| \geq 2, \quad z \in \mathbb{R}^{2d}, \quad t \in [0, T] \quad (2)$$

(Weyl quantization is understood). As a typical model one can consider the case when  $a(t, x, \xi)$  is a quadratic form in  $x, \xi$ , which gives rise to metaplectic operators. Equations of this type turned out to be important in spectral theory [26,27] and in regularization issues for equations with rough coefficients [47]. Depending on the applications, several additional conditions are imposed on the symbol  $a$  and its derivatives. In any case a fundamental problem is to obtain some integral representation of the propagator, from which one can then deduce estimates for the solutions. In principle, *for small time* one expects the propagator to be represented by a Fourier integral operator (FIO)

$$S(t, 0)f = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\Phi(t, x, \eta)} \sigma(t, x, \eta) \hat{f}(\eta) d\eta \quad (3)$$

with a smooth real-valued phase  $\Phi(t, x, \xi)$ , having quadratic growth with respect to the variables  $x, \xi$ , and a symbol  $\sigma(t, \cdot) \in S_{0,0}^0$ , i.e. bounded together with its derivatives. This was first proved in [7,26,27] for a class of symbols  $a(t, x, \xi)$  of polyhomogeneous type, i.e. with an asymptotic expansion in homogeneous terms in  $z = (x, \xi)$ , of decreasing order. These decay conditions are essential for the symbolic calculus to work and the integral representation was in fact constructed by the WKB method. Recently in [10] the above representation (3) was proved to be true for small time only under the assumption (2). Such a representation however does no longer keep valid for large time because of the appearance of caustics; in other terms, the space of FIOs is not an algebra. Different approaches have been proposed by several authors, see e.g. [1,3–5,11,19,21,30,32,35,41,48,49].

In the case of analytic symbols, which is the framework of this paper, there are further technical difficulties and surprisingly, to our knowledge, it is not even known whether the *exact* integral representation (3) holds for an analytic phase and symbol, at least for small time. We will answer positively this question as a byproduct of more general results.

Namely, consider the analytic symbol classes  $S_a^{(k)}$ ,  $k \in \mathbb{N}$ , defined by the estimates

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C^{|\alpha|+|\beta|+1} \alpha! \beta!, \quad |\alpha| + |\beta| \geq k, \quad x, \xi \in \mathbb{R}^d, \quad (4)$$

endowed with the obvious inductive limit topology of Fréchet spaces.

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