

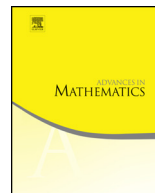


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## Product set phenomena for countable groups

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## ABSTRACT

We develop in this paper novel techniques to analyze local combinatorial structures in product sets of two subsets of a countable group which are “large” with respect to certain classes of (not necessarily invariant) means on the group. Our methods heavily utilize the theory of  $C^*$ -algebras and random walks on groups. As applications of our methods, we extend and quantify a series of recent results by Jin, Bergelson–Furstenberg–Weiss, Beiglböck–Bergelson–Fish, Griesmer and Di Nasso–Lupini to general countable groups.

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## 1. Introduction

## 1.1. General comments

Let  $G$  be a group and let  $\mathcal{L}$ ,  $\mathcal{R}$  and  $\mathcal{S}$  be given sets of subsets of  $G$ . We shall think of  $\mathcal{L}$  and  $\mathcal{R}$  as defining two (possibly different) classes of *large* subsets of the group and the elements of  $\mathcal{S}$  will be regarded as the *structured* subsets of  $G$ .

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In this paper, the term *product set phenomenon* (with respect to the sets  $\mathfrak{L}$ ,  $\mathfrak{R}$  and  $\mathfrak{S}$ ) will refer to the event that whenever  $A \in \mathfrak{L}$  and  $B \in \mathfrak{R}$ , then their *product set*  $AB$ , defined by

$$AB := \{a \cdot b : a \in A, b \in B\}$$

belongs to  $\mathcal{S}$ . If this happens, we shall say that the pair  $(\mathfrak{L}, \mathfrak{R})$  is  $\mathfrak{S}$ -regular.

Perhaps the first occurrence of a (non-trivial) product set phenomenon recorded in the literature is the following classical observation, which is often attributed to Steinhaus (see e.g. [22]): Let  $G$  be a locally compact group with left Haar measure  $m$  and define

$$\mathfrak{L} := \{A \in \mathcal{B}(G) : m(A) > 0\},$$

where  $\mathcal{B}(G)$  denotes the set of Borel sets of  $G$ . Let  $\mathfrak{S}$  denote the set of all subsets of  $G$  with non-empty interior. Then the pair  $(\mathfrak{L}, \mathfrak{L})$  is  $\mathfrak{S}$ -regular, that is to say, the product set of any two Borel sets with positive Haar measures contains a non-empty open set.

## 1.2. Structured sets in countable groups

This paper is concerned with product set phenomena in *countable* groups. To explain our main results, we first need to define what classes of *large* sets and *structured* sets we shall consider. We begin by describing our choices of the structured sets.

Let  $G$  be a countable group. A set  $T \subset G$  is *right thick* if whenever  $F \subset G$  is a finite subset, then there exists  $g$  in  $G$  such that

$$F \cdot g \subset T.$$

We say that a set  $C \subset G$  is *left syndetic* if it has non-trivial intersection with any right thick set, or equivalently, if there exists a finite set  $F \subset G$  such that  $FC = G$ . Let  $\text{Syn}$  denote the set of all left syndetic subsets of  $G$  and define the class of *right piecewise left syndetic sets* PW-Syn by

$$\text{PW-Syn} := \{C \cap T : C \text{ is left syndetic and } T \text{ is right thick}\}.$$

Equivalently, a set  $C \subset G$  is right piecewise left syndetic if there exists a finite set  $F \subset G$  such that the product set  $FC$  is right thick.

A particularly nice sub-class of left syndetic sets is formed by the so called *Bohr sets*. Recall that a set  $C \subset G$  is *Bohr* if there exist a compact Hausdorff group  $K$ , an epimorphism  $\tau : G \rightarrow K$  (a homomorphism with dense image) and a non-empty open set  $U \subset K$  such that

$$C \supset \tau^{-1}(U).$$

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