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Product set phenomena for countable groups

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We develop in this paper novel techniques to analyze local combinatorial structures in product sets of two subsets of a countable group which are "large" with respect to certain classes of (not necessarily invariant) means on the group. Our methods heavily utilize the theory of C*-algebras and random walks on groups. As applications of our methods, we extend and quantify a series of recent results by Jin, Bergelson–Furstenberg–Weiss, Beiglböck–Bergelson–Fish, Griesmer and Di Nasso–Lupini to general countable groups.

1. Introduction

1.1. General comments

Let G be a group and let $\mathfrak{L}$, $\mathfrak{R}$ and $\mathfrak{S}$ be given sets of subsets of G. We shall think of $\mathfrak{L}$ and $\mathfrak{R}$ as defining two (possibly different) classes of *large* subsets of the group and the elements of $\mathfrak{S}$ will be regarded as the *structured* subsets of G.

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In this paper, the term *product set phenomenon* (with respect to the sets $\mathfrak{L}$, $\mathfrak{R}$ and $\mathfrak{S}$) will refer to the event that whenever $A \in \mathfrak{L}$ and $B \in \mathfrak{R}$, then their *product set* AB, defined by

$$AB := \Big\{ a \cdot b \,:\, a \in A,\ b \in B \Big\}$$

belongs to $\mathscr{S}$. If this happens, we shall say that the pair $(\mathfrak{L}, \mathfrak{R})$ is $\mathfrak{S}$-*regular*.

Perhaps the first occurrence of a (non-trivial) product set phenomenon recorded in the literature is the following classical observation, which is often attributed to Steinhaus (see e.g. [22]): Let G be a locally compact group with left Haar measure m and define

$$\mathfrak{L} := \Big\{ A \in \mathscr{B}(G) \,:\, m(A) > 0 \Big\},$$

where $\mathscr{B}(G)$ denotes the set of Borel sets of G. Let $\mathfrak{S}$ denote the set of all subsets of G with non-empty interior. Then the pair $(\mathfrak{L}, \mathfrak{L})$ is $\mathfrak{S}$-regular, that is to say, the product set of any two Borel sets with positive Haar measures contains a non-empty open set.

1.2. Structured sets in countable groups

This paper is concerned with product set phenomena in *countable* groups. To explain our main results, we first need to define what classes of *large* sets and *structured* sets we shall consider. We begin by describing our choices of the structured sets.

Let G be a countable group. A set $T \subset G$ is *right thick* if whenever $F \subset G$ is a finite subset, then there exists g in G such that

$$F \cdot g \subset T.$$

We say that a set $C \subset G$ is *left syndetic* if it has non-trivial intersection with any right thick set, or equivalently, if there exists a finite set $F \subset G$ such that $FC = G$. Let Syn denote the set of all left syndetic subsets of G and define the class of *right piecewise left syndetic sets* PW-Syn by

$$\text{PW-Syn} := \Big\{ C \cap T \,:\, C \text{ is left syndetic and } T \text{ is right thick} \Big\}.$$

Equivalently, a set $C \subset G$ is right piecewise left syndetic if there exists a finite set $F \subset G$ such that the product set FC is right thick.

A particularly nice sub-class of left syndetic sets is formed by the so called *Bohr sets*. Recall that a set $C \subset G$ is *Bohr* if there exist a compact Hausdorff group K, an epimorphism $\tau : G \to K$ (a homomorphism with dense image) and a non-empty open set $U \subset K$ such that

$$C \supset \tau^{-1}(U).$$