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Pseudo-path connected homogeneous continua

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ABSTRACT

The main result of this paper states that every homogeneous *pseudo-path connected* continuum is *weakly chainable*, or equivalently, every homogeneous continuum connected by continuous images of the pseudo-arc is itself a continuous image of the pseudo-arc. We notice that even though there exist homogeneous path connected continua that are not continuous images of an arc (Prajs, 2002), they all are continuous images of the pseudo-arc.

The problem of characterizing of continuous images of the *pseudo-arc* was raised [4] shortly after the hereditary equivalence [16] and the homogeneity [2] of the pseudo-arc were established. Some topologists may consider the pseudo-arc just a peculiar example.

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Yet, the pseudo-arc permeates the fabric of all n-manifolds for $n > 1$. Among compact connected subspaces of manifolds, all but a first category collection are copies of a single homogeneous continuum, the pseudo-arc [3].

Eventually, two authors, A. Lelek [11] and L. Fearnley [7], characterized independently the class of continuous images of the pseudo-arc. Lelek's name for this class, *weakly chainable continua*, was adopted and is still used. A very interesting property is hiding under this modest name. From the beginning it was compared, for good reasons, to local connectivity. Just as local connectivity for compacta: (1) Weak chainability is a continuous invariant. (2) The finite, connected union of weakly chainable continua is weakly chainable. (3) The countable product of weakly chainable continua is weakly chainable. (4) Weakly chainable continua have a common model, the pseudo-arc, just as an arc is a common model for locally connected continua. (5) Just as continuous images of arcs induce partitions of spaces into path components, by (2) weakly chainable continua induce partitions of spaces into *pseudo-path components*, which are, by (1), preserved by continuous maps, and respected by products by (3). In general, this partition is coarser than the partition into usual path-components because there is a continuous surjection from the pseudo-arc onto an arc.

It is remarkable that two most intriguing open problems in planar topology, both about a century old, have major partial solutions involving weakly chainable continua. The planar fixed point problem has been solved for weakly chainable continua by P. Minc [15]. The problem of classifying homogeneous plane continua has been reduced to weakly chainable continua by L.G. Oversteegen and E.D. Tymchatyn [17, Corollary 7, p. 164].

It is known that a path connected compact metric group G is locally connected [9, Theorem 9.68, p. 492], that is, G is a continuous image of an arc. The problem whether each path connected homogeneous continuum is locally connected resisted solution for several decades. Eventually it was solved by the author [18] by providing a counterexample. Here we investigate an analogue question for the *pseudo-path components* (see Definitions 1.6 and 2.1). Is every pseudo-path connected homogeneous continuum weakly chainable? In other words, if X is a homogeneous continuum that is connected by continuous images of the pseudo-arc, must X itself be a continuous image of the pseudo-arc? A positive answer to this question is the main result of this paper.

Our argument is presented in four stages. In the preliminaries, we list definitions and some known properties of the pseudo-arc and weak chainability. We also collect some results on quasi-interiors of sets, which mainly follow the pattern of what is already known in this area. In Section 2 we study properties of pseudo-paths. These results are new, and they attract attention by themselves. Nevertheless we stay within what is needed for the proof of the main result, and carefully avoid building a "general theory of pseudo-paths." More motivation is needed for such a theory. In Section 3 we show that continuous pseudo-fans are weakly chainable. Section 4 contains the proof of the main theorem, which essentially depends on all preceding sections.