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Root statistics of random polynomials with bounded Mahler measure

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A B S T R A C T

The Mahler measure of a polynomial is a measure of complexity formed by taking the modulus of the leading coefficient times the modulus of the product of its roots outside the unit circle. The roots of a real degree N polynomial chosen uniformly from the set of polynomials of Mahler measure at most 1 yield a Pfaffian point process on the complex plane. When N is large, with probability tending to 1, the roots tend to the unit circle, and we investigate the asymptotics of the scaled kernel in a neighborhood of a point on the unit circle. When this point is away from the real axis (on which there is a positive probability of finding a root) the scaled process degenerates to a determinantal point process with the same local statistics (*i.e.* scalar kernel) as the limiting process formed from the roots of complex polynomials chosen uniformly from the set of polynomials of Mahler measure at most 1. Three new matrix kernels appear in a neighborhood of ± 1 which encode information about the correlations between real roots, between complex roots and between real and complex roots. Away from the unit circle, the kernels converge to new limiting kernels, which imply among other things that the expected number of roots in any open subset of $\mathbb{C}$ disjoint from the unit circle converges to a positive number. We also give ensembles with identical statistics drawn from two-dimensional electrostatics with potential theoretic weights, and normal matrices chosen with

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regard to their topological entropy as actions on Euclidean space.

1. Introduction

The study of roots of random polynomials is an old subject extending back at least as far as the early 1930s. Several early results revolve around estimating, as a function of degree N, the number of real roots of polynomials with variously proscribed integer or real coefficients [3,22,23,10,9,6]. More nuanced results about the location of the roots of a degree N random polynomial with i.i.d. coefficients can be specified by an explicit function when integrated over an interval gives the expected number of real roots in that interval [17,18,32,16]. (Such a function is called an *intensity* or *correlation function*, and will be central in the present work.)

A more general, though less precise observation about the locations of roots of random polynomials was made by Erdős and Turán, who prove (in their own paraphrasing)

> ... that the roots of a polynomial are uniformly distributed in the different angles with vertex at the origin if the coefficients "in the middle" are not too large compared with the extreme ones [11].

Strictly speaking, the result of Erdős and Turán is not a result about random polynomials, but rather gives an upper bound, as a function of the coefficients of the polynomials, for the difference between the number of roots in an angular segment of a polynomial from the number assuming radial equidistribution. This can be translated into a result about random polynomials given information about the distribution of coefficients.

Erdős and Turán's result presaged the fact that for many types of random polynomials, the zeros have a tendency to be close to uniformly distributed on the unit circle, at least when the degree is large. One way of quantifying this accumulation is to form a probability measure from a random polynomial by placing equal point mass at each of its roots. This measure is sometimes called the *empirical* measure, and given a sequence of polynomials of increasing degree we can ask whether or not the resulting sequence of empirical measure weak-$*$ converges (or perhaps in some other manner) to uniform measure on the unit circle (or perhaps some other measure). Given a random sequence of such polynomials we can then investigate in what probabilistic manner (almost surely, in probability, etc.) this convergence occurs, if it occurs at all. Another way of encoding convergence of roots to the unit circle (or some other region) can be given by convergence of intensity/correlation functions (assuming such functions exist).

The random polynomials we consider here will *not* have i.i.d. coefficients—a situation first considered rigorously in generality by Hammersley [14]. Our polynomials will be selected uniformly from a certain compact subset of $\mathbb{R}^{N+1}$ (or $\mathbb{R}^N$) as identified with