

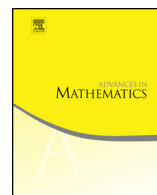


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Generic representation theory of finite fields in nondescribing characteristic[☆]



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ABSTRACT

Let $\text{Rep}(\mathbb{F}; K)$ denote the category of functors from finite dimensional \mathbb{F} -vector spaces to K -modules, where \mathbb{F} is a field and K is a commutative ring. We prove that, if \mathbb{F} is a finite field, and $\text{char } \mathbb{F}$ is invertible in K , then the K -linear abelian category $\text{Rep}(\mathbb{F}; K)$ is equivalent to the product, over all $n \geq 0$, of the categories of $K[GL_n(\mathbb{F})]$ -modules.

As a consequence, if K is also a field, then small projectives are also injective in $\text{Rep}(\mathbb{F}; K)$, and will have finite length. Even more is true if $\text{char } K = 0$: the category $\text{Rep}(\mathbb{F}; K)$ will be semisimple.

In the last section, we briefly discuss ‘ $q = 1$ ’ analogues and consider representations of various categories of finite sets.

The main result follows from a 1992 result by L.G. Kovács about the semigroup ring $K[M_n(\mathbb{F})]$.

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1. Introduction

Let $\mathcal{V}(\mathbb{F})$ be the category of finite dimensional vector spaces over a finite field \mathbb{F} of characteristic p , and let K be a commutative ring, likely a field.

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Then let $\text{Rep}(\mathbb{F}; K)$ denote the category whose objects are functors

$$F : \mathcal{V}(\mathbb{F}) \rightarrow K\text{-modules},$$

and whose morphisms are the natural transformations.

This is a K -linear abelian category in the usual way. For example,

$$0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$$

being short exact in $\text{Rep}(\mathbb{F}; K)$ means that, for any $V \in \mathcal{V}(\mathbb{F})$, the sequence

$$0 \rightarrow F(V) \rightarrow G(V) \rightarrow H(V) \rightarrow 0$$

is a short exact sequence of K -modules.

Our papers [13–17] study the case when $K = \mathbb{F}$. Following terminology used in those papers, we refer to $F \in \text{Rep}(\mathbb{F}; K)$ as a *generic representation* of the field \mathbb{F} . To explain this, note that there are evident connections with the representation theory of the general linear groups $GL_n(\mathbb{F})$, as $F \in \text{Rep}(\mathbb{F}; K)$ defines a family $\{F(\mathbb{F}^n) \mid n = 0, 1, 2, \dots\}$ of $K[GL_n(\mathbb{F})]$ -modules via evaluation. There is even more structure here, as $F(\mathbb{F}^n)$ is a module for the semigroup ring $K[M_n(\mathbb{F})]$, where $M_n(\mathbb{F})$ is the semigroup of all $n \times n$ matrices over \mathbb{F} . Indeed, as described in [14], a generic representation F is roughly the same thing as a compatible sequence of $K[M_n(\mathbb{F})]$ -modules for all n .

There has been much success studying the case when $K = \mathbb{F}$:

- Many extension groups $\text{Ext}_{\text{Rep}(\mathbb{F}; \mathbb{F})}^*(F, G)$ have been calculated when F and G are classical functors. See, e.g., [7,6,24].
- There are connections to both algebraic K -theory and the representation theory of algebraic groups. In particular, both [8,1] show that, if F and G are polynomial functors, then for large n ,

$$\text{Ext}_{\text{Rep}(\mathbb{F}; \mathbb{F})}^k(F, G) \simeq \text{Ext}_{\mathbb{F}[GL_n(\mathbb{F})]}^k(F(\mathbb{F}^n), G(\mathbb{F}^n)).$$

- There is a deep connection between $\text{Rep}(\mathbb{F}_p; \mathbb{F}_p)$ and the category of unstable modules over the mod p Steenrod algebra of algebraic topology. See [11,21,13,15].

By contrast, there has been relatively little said about the structure of $\text{Rep}(\mathbb{F}; K)$ when K is a field of characteristic different than p . Our main theorem here remedies this.

Theorem 1.1. *Let \mathbb{F} be a finite field of characteristic p . If p is invertible in a commutative ring K , there is a natural equivalence of K -linear abelian categories*

$$\text{Rep}(\mathbb{F}; K) \simeq \prod_{n=0}^{\infty} K[GL_n(\mathbb{F})]\text{-modules}.$$

Some structural results about $\text{Rep}(\mathbb{F}; K)$ are immediate corollaries.

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