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Generic representation theory of finite fields in nondescribing characteristic ☆

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A R T I C L E   I N F O

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A B S T R A C T

Let $\operatorname{Rep}(\mathbb{F}; K)$ denote the category of functors from finite dimensional $\mathbb{F}$-vector spaces to K-modules, where $\mathbb{F}$ is a field and K is a commutative ring. We prove that, if $\mathbb{F}$ is a finite field, and $\operatorname{char} \mathbb{F}$ is invertible in K, then the K-linear abelian category $\operatorname{Rep}(\mathbb{F}; K)$ is equivalent to the product, over all $n \geq 0$, of the categories of $K[GL_n(\mathbb{F})]$-modules.

As a consequence, if K is also a field, then small projectives are also injective in $\operatorname{Rep}(\mathbb{F}; K)$, and will have finite length. Even more is true if $\operatorname{char} K = 0$: the category $\operatorname{Rep}(\mathbb{F}; K)$ will be semisimple.

In the last section, we briefly discuss '$q = 1$' analogues and consider representations of various categories of finite sets.

The main result follows from a 1992 result by L.G. Kovács about the semigroup ring $K[M_n(\mathbb{F})]$.

1. Introduction

Let $\mathcal{V}(\mathbb{F})$ be the category of finite dimensional vector spaces over a finite field $\mathbb{F}$ of characteristic p, and let K be a commutative ring, likely a field.

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Then let $\mathrm{Rep}(\mathbb{F}; K)$ denote the category whose objects are functors

$$F : \mathcal{V}(\mathbb{F}) \to K\text{-modules},$$

and whose morphisms are the natural transformations.

This is a K-linear abelian category in the usual way. For example,

$$0 \to F \to G \to H \to 0$$

being short exact in $\mathrm{Rep}(\mathbb{F}; K)$ means that, for any $V \in \mathcal{V}(\mathbb{F})$, the sequence

$$0 \to F(V) \to G(V) \to H(V) \to 0$$

is a short exact sequence of K-modules.

Our papers [13–17] study the case when $K = \mathbb{F}$. Following terminology used in those papers, we refer to $F \in \mathrm{Rep}(\mathbb{F}; K)$ as a *generic representation* of the field $\mathbb{F}$. To explain this, note that there are evident connections with the representation theory of the general linear groups $GL_n(\mathbb{F})$, as $F \in \mathrm{Rep}(\mathbb{F}; K)$ defines a family $\{F(\mathbb{F}^n) \mid n = 0, 1, 2, \ldots\}$ of $K[GL_n(\mathbb{F})]$-modules via evaluation. There is even more structure here, as $F(\mathbb{F}^n)$ is a module for the semigroup ring $K[M_n(\mathbb{F})]$, where $M_n(\mathbb{F})$ is the semigroup of all $n \times n$ matrices over $\mathbb{F}$. Indeed, as described in [14], a generic representation F is roughly the same thing as a compatible sequence of $K[M_n(\mathbb{F})]$-modules for all n.

There has been much success studying the case when $K = \mathbb{F}$:

- Many extension groups $\mathrm{Ext}^*_{\mathrm{Rep}(\mathbb{F};\mathbb{F})}(F, G)$ have been calculated when F and G are classical functors. See, e.g., [7,6,24].
- There are connections to both algebraic K-theory and the representation theory of algebraic groups. In particular, both [8,1] show that, if F and G are polynomial functors, then for large n,

$$\mathrm{Ext}^k_{\mathrm{Rep}(\mathbb{F};\mathbb{F})}(F, G) \simeq \mathrm{Ext}^k_{\mathbb{F}[GL_n(\mathbb{F})]}\big(F\big(\mathbb{F}^n\big), G\big(\mathbb{F}^n\big)\big).$$

- There is a deep connection between $\mathrm{Rep}(\mathbb{F}_p; \mathbb{F}_p)$ and the category of unstable modules over the mod p Steenrod algebra of algebraic topology. See [11,21,13,15].

By contrast, there has been relatively little said about the structure of $\mathrm{Rep}(\mathbb{F}; K)$ when K is a field of characteristic different than p. Our main theorem here remedies this.

Theorem 1.1. *Let $\mathbb{F}$ be a finite field of characteristic p. If p is invertible in a commutative ring K, there is a natural equivalence of K-linear abelian categories*

$$\mathrm{Rep}(\mathbb{F}; K) \simeq \prod_{n=0}^{\infty} K\big[GL_n(\mathbb{F})\big]\text{-modules.}$$

Some structural results about $\mathrm{Rep}(\mathbb{F}; K)$ are immediate corollaries.