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Random Latin squares and 2-dimensional expanders

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A B S T R A C T

Expander graphs have been playing an important role in combinatorics and computer science over the last four decades. In recent years a theory of high dimensional expanders is emerging, but as of now all known examples of expanders (random and explicit) have unbounded degrees. The question of existence of bounded degree high dimensional expanders was raised by Gromov and by Dotterer and Kahle. In this paper we present a new model, based on Latin squares, of 2-dimensional complexes of bounded edge degrees that are expanders with probability tending to 1.

1. Introduction

The notion of expander graphs has been extremely useful in computer science, combinatorics and even pure mathematics (see [10,14] and the references therein). In recent years there is a growing interest in high-dimensional expanders (see the survey [15]). The

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k-dimensional version of the graphical Cheeger constant, called "coboundary expansion", came up independently in the work of Linial, Meshulam and Wallach [12,18] on homological connectivity of random complexes and in Gromov's remarkable work [7,3] where it is shown that this notion of expansion implies the topological overlap property (see Section 6).

We recall some topological terminology. Let X be a simplicial complex on the vertex set V. For $k \geq 0$, let $X^{(k)}$ denote the k-dimensional skeleton of X and let $X(k)$ be the family of k-dimensional faces of X. Let $D_k(X)$ be the maximum number of $(k+1)$-dimensional faces of X containing a common k-dimensional face. Let $C^k(X;\mathbb{F}_2)$ denote the space of $\mathbb{F}_2$-valued k-cochains of X. The k-coboundary map $d_k : C^k(X;\mathbb{F}_2) \to C^{k+1}(X;\mathbb{F}_2)$ is given

$$d_k\phi(v_0,\cdots,v_{k+1}) = \sum_{i=0}^{k+1} \phi(v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_{k+1}).$$

It will be convenient to augment the cochain complex $\{C^i(X;\mathbb{F}_2)\}_{i=0}^{\infty}$ with a (-1)-degree term $C^{-1}(X;\mathbb{F}_2) = \mathbb{F}_2$ with a coboundary map $d_{-1} : C^{-1}(X;\mathbb{F}_2) \to C^0(X;\mathbb{F}_2)$ given by $d_{-1}(a)(v) = a$ for $a \in \mathbb{F}_2$, $v \in V$.

Let $Z^k(X;\mathbb{F}_2) = \ker d_k$ be the space of $\mathbb{F}_2$-valued k-cocycles of X and let $B^k(X;\mathbb{F}_2) = d_{k-1}(C^{k-1}(X;\mathbb{F}_2))$ be the space of $\mathbb{F}_2$-valued k-coboundaries of X. The k-th reduced cohomology group of X with $\mathbb{F}_2$ coefficients is

$$\tilde{H}^k(X;\mathbb{F}_2) = \frac{Z^k(X;\mathbb{F}_2)}{B^k(X;\mathbb{F}_2)}.$$

For $\phi \in C^k(X;\mathbb{F}_2)$, let $[\phi]$ denote the image of ϕ in the quotient space $C^k(X;\mathbb{F}_2)/B^k(X;\mathbb{F}_2)$. Let

$$\|\phi\| = \left|\left\{\sigma \in X(k) : \phi(\sigma) \neq 0\right\}\right| = \left|\mathrm{supp}(\phi)\right|$$

and

$$\left\|[\phi]\right\| = \min\left\{\left|\mathrm{supp}(\phi + d_{k-1}\psi)\right| : \psi \in C^{k-1}(X;\mathbb{F}_2)\right\}.$$

We will sometimes write $\|\phi\|_X$ in case of ambiguity concerning X.

Definition. The *k-th coboundary expansion constant* of X (see [12,18,7,2]) is defined by

$$h_k(X) = \min\left\{\frac{\|d_k\phi\|}{\|[\phi]\|} : \phi \in C^k(X;\mathbb{F}_2) - B^k(X;\mathbb{F}_2)\right\}.$$

A complex X is a (k,d,ϵ)-*expander* if

$$D_{k-1}(X) \leq d \quad \text{and} \quad h_{k-1}(X) \geq \epsilon.$$