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Deformation quantization of Leibniz algebras

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ABSTRACT

In this paper, we use the local integration of a Leibniz algebra $\mathfrak{h}$ using a Baker–Campbell–Hausdorff type formula in order to deformation quantize its linear dual $\mathfrak{h}^*$. More precisely, we define a natural rack product on the set of exponential functions on $\mathfrak{h}^*$ which extends to a rack action on $\mathcal{C}^\infty(\mathfrak{h}^*)$.

0. Introduction

In this paper, we solve an old problem in symplectic geometry, namely we propose a way how to quantize the dual space of a Leibniz algebra $\mathfrak{h}$. This dual space $\mathfrak{h}^*$ is some kind of generalized Poisson manifold, as the bracket of $\mathfrak{h}$ is not necessarily skew-symmetric. Intimately linked to this question is the integration of Leibniz algebras.

In the search of understanding the periodicity in K-theory, J.-L. Loday introduced Leibniz algebras as non-commutative analogues of Lie algebras. More precisely, a real

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Leibniz algebra is a real vector space with a bracket which satisfies the (left) Leibniz identity

$$\big[X,[Y,Z]\big] = \big[[X,Y],Z\big] + \big[Y,[X,Z]\big],$$

but is not necessarily skew-symmetric. Leibniz algebras are a well-established algebraic structure generalizing Lie algebras (those Leibniz algebras where the bracket is skew-symmetric) with their own structure, deformation and homology theory. In the same way the Lie algebra homology of matrices (over a commutative ring containing the rational numbers) defines additive K-theory (i.e. cyclic homology), the Leibniz homology of matrices defines some non-commutative additive K-theory (in fact, Hochschild homology). Loday was mainly interested in the properties of the corresponding homology theory on "group level" ("Leibniz K-Theory"), and therefore asked the question: which (generalization of the structure of Lie groups) is the correct structure to integrate Leibniz algebras?

Kinyon [15] explored Lie racks as a structure integrating Leibniz algebras. Racks are roughly speaking an axiomatization of the structure of the conjugation in a group. The rack product on a group is simply given by

$$g \rhd h := ghg^{-1},$$

and a general rack product on a set X is a binary operation satisfying for all $x, y, z \in X$ that $x \rhd - : X \to X$ is bijective and the autodistributivity relation

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z).$$

Lie racks are the smooth analogues of racks. Kinyon showed (see Theorem 1.25) that the tangent space at the distinguished element 1 of a Lie rack carries in a natural way a Leibniz bracket. The idea is to differentiate twice the rack structure, mimicking exactly how the conjugation in a Lie group is differentiated to give first the map Ad, the adjoint action of the group on the Lie algebra, and then the Lie bracket in terms of ad, the adjoint action of the Lie algebra on itself. He did not judge racks to be the correct objects integrating Leibniz algebras. As a reason for this, he showed that all Leibniz algebras integrate into Lie racks, but in a kind of arbitrary way, as this integration does not appear to give Lie groups in case one started with a Lie algebra. It is clear (and useful as a guiding principle) that from this point of view, integrating Leibniz algebras means just an *integration of the adjoint action* of a Leibniz algebra on itself. From here stems the most important example for us of a rack product, namely

$$X \rhd Y := e^{\operatorname{ad}_X}(Y),$$

for all $X, Y \in \mathfrak{h}$ for a Leibniz algebra $\mathfrak{h}$.

On the other hand, Covez [10] showed in his 2010 doctoral thesis how to adapt the homological proof of Lie's Third Theorem to Leibniz algebras. Regarding a given real